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TREATISE

ON THE

INTEGRAL CALCULUS.

211

CONTAINING

THE INTEGRATION OF
EXPLICIT FUNCTIONS OF ONE VARIABLE;

TOGETHER WITH

THE THEORY OF DEFINITE INTEGRALS
AND OF ELLIPTIC FUNCTIONS.

BY J. HYMERS, M.A.

FELLOW AND TUTOR OF ST. JOHN'S COLLEGE, CAMBRIDGE.

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CONTENTS.

SECTION I.

ELEMENTARY INTEGRALS.

PAGE

ART. 1. MEANING and object of the Integral Calculus. 2. Explanation of the terms and notation employed. 3—5. General statement of the various methods of integration, and of the quantities by which integrals are expressed.......... 1

6. To every integral an arbitrary constant must be added. 7. Constant factors, which multiply differential coefficients, may be written outside the sign of integration.............. 4

8. The integral of the sum or difference of several differential coefficients, is equal to the sum or difference of the integrals of these differential coefficients. 9. Investigation of the formula $\int_x U d_x V = UV - \int_x V d_x U$.

10. $\int_x \frac{d_x U}{V} = \frac{U}{V} + \int_x \frac{U}{V^2} d_x V$............... 5

11. The integral of any power of x is formed by adding unity to the index, and dividing by the increased index.... 6

12. Generally, $\int_x a u^{m-1} d_x u = \frac{a u^m}{m} + C$; examples of this formula. 13. Instances of artifices by which expressions are reduced to the above form.................................. 7

14. $\int_x \frac{1}{x} = \log_\epsilon x + C$. 15. The integral of any fraction whose numerator is the differential coefficient of the denominator, or bears an invariable ratio to it, is the Naperian logarithm of the denominator, or bears that same ratio to it. Examples... 10

PAGE

16. Hence all rational integral algebraic functions of one variable can be integrated.................................. 11

17, 18. Formulæ for changing the hypothesis of integration, $\int_x \{f(z) d_x z\} = \int_z f(z)$, $\int_x u = \int_z u\, d_z x$,.............. 12

19, 20. The integral $\int_x (a + b x^n)^{\frac{p}{q}} x^{m-1}$ can be reduced to the sum of a series of terms of the form $c\int_z z^r$, if $\frac{m}{n} =$ a positive integer, or $\frac{m}{n} + \frac{p}{q} =$ a negative integer. Examples........... 13

21. Integration of elementary logarithmic forms.......... 16

22. Integration of elementary circular forms. 23. Integration of certain other logarithmic and circular forms of great use.. 20

24. The above circular and logarithmic forms may be deduced one from the other 21

25, 26. Application of the logarithmic and circular forms to integrate a variety of expressions......................... 22

27. In some cases it is more convenient to reduce the expression to the proper form by substitution. Examples. 28. Instances of common artifices by which expressions are reduced to the elementary forms........................... 25

SECTION II.

RATIONAL FRACTIONS.

Art. 29. The dimension of the numerator is less, at least by unity, than that of the denominator. 30. Proof of the possibility of resolving any given rational fraction into partial fractions. 31. Integration of every rational fraction reducible to that of four forms.................................... 32

33. Determination of the partial fractions corresponding to the simple factors which occur only once in the denominator. Examples. 34. Second method, by which the division of the denominator by the factors is avoided....... 35

PAGE

35. Determination of the groups of partial fractions corresponding to the simple factors each of which occurs more than once in the denominator. Examples. 36. Method of finding, immediately, any single fraction in a group........ 39

37. Determination of the partial fractions corresponding to the irreducible quadratic factors which occur only once in the denominator. Examples. 38. Second method, by which the division of the denominator by the factors is avoided.. 42

39. Determination of the groups of partial fractions corresponding to the irreducible quadratic factors, each of which occurs more than once in the denominator. Examples......... 45

Particular methods of integrating the following fractions, with examples:

40. $\frac{1}{1-x^n}$. 41. $\frac{1}{1+x^n}$. 42. $\frac{x^m}{x^n \pm 1}$. 43. $\frac{x^m}{a+bx^n}$ 47

44. $\frac{x^{m-1} \pm x^{n-m-1}}{x^n+1}$. 45. $\frac{x^{m-1} \pm x^{n-m-1}}{x^n-1}$ 54

46. $\frac{x^m}{x^{2n} - 2\cos\theta\, x^n + 1}$. 48. $\frac{x^m}{a+bx^n+cx^{2n}}$ 56

SECTION III.

RATIONALIZATION.

Art. 49. Every integral must be considered as found when it can be reduced to the form $\int_z R(z)$, (where $R(z)$ denotes a rational function of z) or to such another irrational form as admits of being rationalized........................ 59

Methods of rationalizing $\int_x u$ for the following values of u.

I. $R\left\{x^r, \left(\frac{a+bx}{a_1+b_1x}\right)^{\frac{m}{n}}, \left(\frac{a+bx}{a_1+b_1x}\right)^{\frac{p}{q}}, \&c.\right\}$

II. $R(x, \sqrt{bx \pm cx^2})$.

III. $R(x, \sqrt{a+bx+cx^2})$ 61

PAGE

IV. $R(x, \sqrt{a+bx}, \sqrt{a_1+b_1x})$.

V. $x^m.R(x^n, \sqrt{a+b^2x^{2n}}, bx^n \pm \sqrt{a+b^2x^{2n}})$

where $\frac{m+1}{n}$ is an integer.................... 63

VI. $x^{m-1}(a+bx^n)^{\frac{p}{q}} R(x^n)$ when $\frac{m}{n}$ or $\frac{m}{n}+\frac{p}{q}$ is a positive or negative integer 64

50. Instances of rationalizing integrals by particular assumptions 64

SECTION IV.

FORMULÆ OF REDUCTION.

Art. 51, 52. General explanation of the method........ 65

53. Application to the integral $\int_x x^{m-1}(a+bx^n)^p$, or $\int_x x^{m-1}X^p$, which can always be made to depend upon another of the same form, in which one of the indices is altered, viz. that of x by the addition or subtraction of n, and that of X by the addition or subtraction of unity.................. 66

54, 55. Rule for the assumption. Examples............ 67

56. Cases of failure. 57. In some cases it is more advantageous to integrate by parts............................ 74

58—60. Application of the method to the more general case $\int_x x^{m-1}(a+bx^n+cx^{2n})^p$. Examples.................. 77

61. Extension of the method to the general case

$\int_x x^{m-1}(a+bx^n+cx^{2n}+\&c.+tx^{kn})^p$ 82

SECTION V.

EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

Art. 62. Fundamental formula $\int_x a^u d_x u = \frac{1}{\log_\epsilon a} . a^u + C$. 63. Integration of some expressions of the form $\int_x f(a^x)$. 64. The expression $e^x u$ can be integrated whenever u is reducible to the form $y+d_x y$. 65—67. Integration of $a^x . R(x)$...... 83

PAGE

68. Integration of expressions involving logarithmic functions of the variable. Examples of the form $\int_x u \log_\epsilon v$. 69, 70. Integration of $u(\log_\epsilon x)^n$, $u(\log_\epsilon x)^{-n}$, u denoting an algebraic function of x 86

SECTION VI.

CIRCULAR FUNCTIONS.

Art. 71. Fundamental formulæ, $\int_x \cos u \, d_x u = \sin u$, $\int_x \sin u \, d_x u = -\cos u$, $\int_x (\sec u)^2 d_x u = \tan u$, &c. 73. Other elementary integrals, such as $\int_x \frac{d_x u}{\sin u} = \log_\epsilon \left(\tan \frac{u}{2}\right)$,

$\int_x \frac{d_x u}{\cos u} = \log_\epsilon \tan \left(\frac{\pi}{4} + \frac{u}{2}\right)$, &c. 90

74. The expression $(\sin x)^m (\cos x)^n$ is immediately integrable, if either m or n be a positive odd integer, or $m+n$ a negative even integer. 75. In other cases $\int_x (\sin x)^m (\cos x)^n$ must be found by the method of reductions; rule for the assumption. Examples. 76. In some cases $\int_x (\sin x)^m (\cos x)^n$ is more conveniently reduced by integration by parts......... 92

77. Expressions for $\int_x (\sin x)^n$, $\int_x (\cos x)^n$, in sines and cosines of multiple angles 96

78—83. Integration of the following expressions:

$\sin mx \cos nx$, $\cos(mx + \alpha)\cos(nx + \beta)$,

$\frac{1}{a + b\cos x}$, $\frac{1}{a + b\sin x}$, $\frac{1}{(a + b\cos x)^n}$ 98

84—86. Integration of the expressions $e^{ax}\sin mx$, $e^{ax}\cos mx$, $e^{ax}(\sin x)^n$, $e^{ax}(\cos x)^n$. 87. All functions of $\sin x$ and $\cos x$ may be converted into exponential functions 102

88—91. Integration of the expressions $e^{ax}(\sin x)^n(\cos x)^m$, $u\sin^{-1}v$, $u\tan^{-1}v$, $x^n\cos x$, $x^n\sin x$, $(\sin^{-1}x)^n$, $x^n(\cos x)^m$ 105

92. Certain integrals, involving circular functions, which occur in the application of Mathematics to Natural Philosophy. 93. The integration of algebraic expressions is sometimes facilitated by converting them into circular functions 108

SECTION VII.

INTEGRATION BETWEEN LIMITS, AND BY INFINITE SERIES.

PAGE

ART. 94. Definition of a definite integral. 95. Explanation by reference to the areas of curves. 96. The definite integral of u between the limits $x=a$, $x=b$, is written $(\int_{x=b} - \int_{x=a}) u$. 97. Illustration by finding the area of different portions of a circle 111

98. Examples of deducing the value of the definite integral from the indefinite one. Value of $(\int_{x=\frac{\pi}{2}} - \int_{x=0}) (\sin x)^n$, and those of others that can be reduced to it. 99. Properties of the definite integral, when the quantity under the sign has periodical values. 100. Division of a definite integral into intervals 115

101. Statement of the principal methods of finding definite integrals in cases when the indefinite ones cannot be obtained, with the following exemplifications.

102. $(\int_{t=\infty} - \int_{t=0}) e^{-t^2} = \frac{1}{2}\sqrt{\pi}$.

103. $(\int_{x=1} - \int_{x=0}) \left(\log_e \frac{1}{x}\right)^{-\frac{1}{2}} = \sqrt{\pi}$. 124

104. $(\int_{x=\infty} - \int_{x=0}) \frac{x^{m-1}}{1+x^n} = \frac{\pi}{n \sin \frac{m\pi}{n}}$.

105. $(\int_{x=1} - \int_{x=0}) \frac{x^{m-1} - x^{n-m-1}}{1-x^n} = \frac{\pi}{n \tan \frac{m\pi}{n}}$ 126

106. $(\int_{x=\infty} - \int_{x=0}) \frac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}} \sin mx = \frac{1}{2} \frac{e^m - e^{-m}}{e^m + 2\cos a + e^{-m}}$ 127

107. Properties of the first of Euler's Integrals

$(\int_{x=1} - \int_{x=0}) \frac{x^{p-1}}{(1-x^n)^{\frac{n-q}{n}}}$ 128

PAGE

108. Properties of the 2nd of Euler's Integrals

$$\left(\int_{x\mp 1} - \int_{x=0}\right)\left(\log_\epsilon \frac{1}{x}\right)^{n-1} = \Gamma(n),$$

$\Gamma(n+1) = n\,\Gamma(n)$, $\Gamma(m) . \Gamma(1-m) = \dfrac{\pi}{\sin m\pi}$ 130

109. Employment of impossible quantities in finding definite integrals. Example, $\left(\int_{x=\infty} - \int_{x=0}\right) x^p e^{-qx} \cos rx$ 131

110. Method of differentiating or integrating under the sign of definite integration. 111. Its use in deducing values of definite integrals from known results. 112. Application of it to find the values of $\left(\int_{x=\infty} - \int_{x=0}\right) e^{-a^2x^2} \cos rx$;

$$\left(\int_{x=\infty} - \int_{x=0}\right) e^{-x^2 - \frac{c^2}{x^2}};\ \left(\int_{x=\infty} - \int_{x=0}\right) \frac{\cos cx}{1+x^2} \text{ ....} \quad 132$$

113. Bernouilli's theorem for finding the value of an integral in a series. 114. To find $\left(\int_{x=a+b} - \int_{x=a}\right) u$ in a series ascending by powers of b. 115. More accurate determination by dividing the difference between the limits into (n) equal intervals. 116. Proof that $\left(\int_{x=b} - \int_{x=a}\right) u =$ limit of $(h u_{x=a} + h u_{x=a+h} + \&c. + h u_{x=b-h})$. 117, 118. Other approximations to the value of $\left(\int_{x=b} - \int_{x=a}\right) u$. 119. Geometrical explanation of these series 137

120. Expansions of $\log_\epsilon(1+x)$, $\tan^{-1} x$, &c. found by integration. 121. Value of $\int_x (a+bx^n)^{\frac{p}{q}} x^{m-1}$ in a series. 122, 123. General values of $\int_x \frac{1}{\sqrt{1-c^2(\sin x)^2}}$, $\int_x \sqrt{1-c^2(\sin x)^2}$, and also between the limits $x=0$, $x=\frac{1}{2}\pi$. 124, 125. More convergent series for the above integrals. 126. Instances of other integrals that can be reduced to them 141

127. To find $\int_x \log_\epsilon(1+n\cos x)$ in a series. 128. Values of definite integrals deduced from it. 129, 130. To find $\int_x (1+e\cos x)^n$ in a series. 131, 132. From the coefficients of the expansion of $(1+e\cos x)^{-n}$ to deduce those of the expansion of $(1+e\cos x)^{-n-1}$.............................. 147

133. To find $\int_t e^{-t^2}$ between the limits $t=0$, $t=t_1$, in a series.. 151

SECTION VIII.

AREAS, LENGTHS, AND VOLUMES.

PAGE

ART. 134, 135. General expression for the area and length of a curve referred to rectangular co-ordinates; and for the volume and area of the surface of the figure generated by it. 136. If the tangent to a curve make an angle (θ) with the ordinate, and p be the length of the perpendicular dropped upon it from the origin, the length of the arc $= \pm (d_\theta p + \int_\theta p)$. Hence we can find a curve whose arc shall represent any proposed integral 152

137. Examples. (1) Area of a circle. (2) Area of an ellipse. (3) Length of a circular arc. (4, 5) Surface and volume of any portion of a sphere. (6) Volume of a spheroid. (7) Length of a parabolic arc. (8) Area of a parabolic segment cut off by any ordinate. (9, 10) Volume and surface of a parabolic frustum. (11) Length of an elliptic arc. (12) Determination of two arcs whose difference is equal to an assignable straight line. (13) Division of an elliptic quadrant into two parts whose difference shall be equal to the difference of the semiaxes. (14) Length of a Hyperbolic arc. (15) Difference between the lengths of the asymptote and infinite hyperbolic arc in a series. (16, 17) Surface of prolate and oblate spheroid. (18, 19) Area and length of cycloid. (20, 21) Surface and volume of the figures generated by a cycloidal area revolving about its axis or base. (22) Length of the Epitrochoid. (23) Area of catenary. (24) Area and volume of Cissoid of Diocles. (25) Volume of figure generated by the revolution of a Conchoid about its asymptote........... 154

138. Formulæ for Polar co-ordinates. Examples. (1) Area of Conchoid. (2) Area of sector of Hyperbola bounded by straight lines passing through the center. (3) Area of Hyperbolic sector contained by lines through the focus. (4) Area and length of the Lemniscata. (5) Area of curve traced out by the intersection of normals to an ellipse at right angles to one another. (6) Area and length of Hypocycloid. (7) Area and length of the involute of a circle. 139. To find the area of the Nodus of a curve; area and length of the evolute of an ellipse 170

PAGE

140. Instance of finding the volume when the revolving area is bounded by two curves. 141. Instances of finding the volume of a figure generated by the motion of a variable area parallel to itself: (1) Volume of any pyramid. (2) Volume of any segment of a paraboloid. (3) Volume and surface of a groin 178

142. If in a curvilinear area, there be drawn an odd number of equidistant ordinates, we may obtain an approximation to the value of the area, by adding the first and last ordinates to four times the sum of the even and twice the sum of the odd ones, and multiplying by ⅓rd the common distance of the ordinates 181

SECTION IX.

ELLIPTIC FUNCTIONS.

Art. 143, 144. Elementary forms of elliptic functions; explanation of the notation used. 145. All integrals of the form $\int_x \frac{R(x)}{\sqrt{X}}$, where X denotes a polynomial of the fourth order, can be reduced to this class 182

146. Three principal steps in the analysis of $\int_x R(x)$. 147, 148. To transform $\int_x \frac{R(x)}{\sqrt{X}}$ into $\int_z \frac{R(z^2)}{\sqrt{(f+gz^2)(h+kz^2)}}$. 149—151. To transform $\int_z \frac{R(z^2)}{\sqrt{(f+gz^2)(h+kz^2)}}$ into $\int_\theta \frac{R\{(\sin\theta)^2\}}{\sqrt{1-c^2(\sin\theta)^2}}$ 184

152—154. To investigate formulæ of reduction for $\int_\theta \frac{R\{(\sin\theta)^2\}}{\sqrt{1-c^2(\sin\theta)^2}}$. 155. Particular reduction of $\int_x \frac{f+gx^2}{\sqrt{\alpha^2+2\alpha\beta\cos\alpha x^2+\beta^2x^4}}$ to the elementary forms 190

PAGE

156—158. Formulæ for the comparison of elliptic functions of the first order. 159, 160. Amplitude of a function which is equal to the sum or difference of two given functions 198

161. The amplitudes σ, ϕ, ψ, which satisfy the equation $F(\sigma) = F(\phi) + F(\psi)$, may be represented by the sides of a spherical triangle. 162, 163. Consequent verification of preceding results. 164, 165. To find the amplitude of a function which is a given multiple or part of another function. 166. Application of the foregoing formulæ to the arcs of the Lemniscata 201

167—171. Formulæ for the comparison of elliptic functions of the second order. 172. Application to elliptic arcs. 173. Corresponding properties of hyperbolic arcs. 174. The arc of the curve which is constantly touched by a perpendicular to the diameter of an ellipse through its extremity, represents an elliptic function of the first order 205

175, 176. To approximate to the values of the complete functions F_c, E_c, when $c = 1$ very nearly. 177—179. Lagrange's transformation of a function of the first order into another similar function with a different modulus and amplitude. 180—182. Consequent approximations to the value of $F_c(\phi)$, according as $c > \sin 45^\circ$, or $< \sin 45^\circ$ 213

183. Lagrange's transformation applied to functions of the second order. 184. Formula which connects three consecutive functions of the second order. 185. An elliptic function of the first order can be expressed by two of the second order. Landen's theorem for expressing a hyperbolic arc by two elliptic arcs. 186. Difference of asymptote and infinite hyperbolic arc expressed by difference between two elliptic quadrants 220

187. Approximate value of $G_c(\phi) = \int_\phi \frac{a_0 + b_0 (\sin\phi)^2}{\Delta_c(\phi)}$. 188. Approximate value of the length of any elliptic or hyperbolic arc 223

189—191. Abel's notation for expressing the amplitude in terms of the elliptic function, and its uses. 192. Enunciation of Jacobi's first theorem for transforming $F_k(\phi)$ into a

PAGE

similar function with a smaller modulus. 193—199. Its demonstration, which is founded upon the analogy which exists between an elliptic function and an angle. 200. Method of computing the new amplitude. 201. Ivory's extension of the theorem to the case of even numbers 225

202, 203. Jacobi's second theorem for transforming $F_k(\phi)$ into a similar function with a larger modulus. 204, 205. Its demonstration, and use. 206. Remarkable relation between the complete functions. 207. Hence we can find any multiple or aliquot part of an elliptic function. 208, 209. Lagrange's and Legendre's transformations are particular cases of Jacobi's, corresponding to $p=2$ and $p=3$. 237

210. Proof of Legendre's property of complete complementary functions. 211. Differential equations of the second order which determine F_c, E_c, in terms of b or c 242

212, 213. Properties of functions of the third order; the parameter may be always supposed less than the modulus. 214. A function with a negative parameter not intermediate to 0 and $-c^2$, can be made to depend upon one with a parameter between these limits. 215. A function with a positive parameter can be made to depend upon one with a parameter intermediate to -1 and $-c^2$. 216. There are only two independent forms of the parameter 244

217, 218. Formulæ for the comparison of functions of the third order. 219—221. Complete functions of the third order can always be expressed by functions of the first and second orders. 222. Case of imaginary parameters 249

INTEGRAL CALCULUS.

SECTION I.

ELEMENTARY INTEGRALS.

ART. 1. THE Integral Calculus is the reverse of the Differential, and has for its object to determine the value of a function the differential coefficient of which is known, in the same manner as the object of the latter is to determine the differential coefficient when the function itself is given; or, more generally, the object of the Integral Calculus is to discover the relations which exist between the variables and their functions, from given equations between the variables, the functions, and the differential coefficients of the functions.

Hence, our object is to determine y in terms of x, or the relation which exists between them, from the equations

$$d_x y = f(x),$$

$$\text{or } f(d_x^n y,\ d_x^{n-1} y, \ldots x,\ y) = 0;$$

or to assign the relation between x, y, z, (where z is a function of x and y), or between a greater number of variables and their functions, from the equation

$$f(x,\ y,\ z,\ d_x z,\ d_y z) = 0,$$

or from other equations in which a greater number of variables and differential coefficients of higher orders are involved.

2. We shall begin with the solution of the simplest case

$$d_x y = u \text{ (a function of } x),$$

in which it is required to find the value of a function of one variable, when the first differential coefficient is given explicitly in terms of that variable; y representing the unknown function, and u the given differential coefficient.

The required function y is usually expressed by $\int_x u$, ($\int_x$ being the symbol of an operation precisely the reverse of that indicated by d_x in the Differential Calculus) and is called the *integral* of u with respect to x. Hence if $\int_x$ and d_x be prefixed to the same function, they neutralize one another, that is,

$$\int_x (d_x u) = u.$$

The operation by means of which the integral of a given differential coefficient is determined, is called *Integration*; to *integrate* a differential coefficient, is to find its integral, that is, the function from which it is derived. Hence $\int_x u$ means a function of x, whose first differential coefficient is u, or such that the limit of the ratio which its increment bears to the corresponding increment of x, (when the latter is indefinitely diminished) is u. All functions of x which are proposed for integration, are looked upon as the first differential coefficients of certain unknown functions, which we are required to find.

Again, as the same function admits of successive differentiations, so a function may be integrated any number of times; and as $d_x^2 u$ means $d_x (d_x u)$, so $\int_x^2 u$ means $\int_x (\int_x u)$, and is called the second integral of u with respect to x; and $\int_x^n u$ is called the n^{th} integral of u with respect to x.

3. The above definitions and notation being understood, we proceed to deduce the rules for integration. Every rule given in the Differential Calculus for finding the differential coefficient of a function of one variable, being inverted, will furnish a corresponding rule for integration.

Thus the Integral Calculus, at least in the simpler parts of the subject, requires no new investigation of principles, but depends for them entirely upon the Differential Calculus; and to a person who is familiar with the latter, it offers few difficulties beyond those arising from complicated algebraical operations. Expertness in performing these, and in foreseeing to what result any substitution will lead, is very necessary in this subject; since, with all the rules that can be given, the integration of many formulæ may be facilitated, and sometimes can only be effected by particular transformations and artifices, which the student must himself discover.

4. The problem of integrating a given differential coefficient, may be resolved into these two grand divisions.

I. To find the values of the *elementary integrals*, that is, such as are not capable of being transformed into simpler expressions; as for instance

$$\int_x x^m, \quad \int_x \frac{1}{x}, \quad \int_x \frac{1}{a^2 + x^2}, \quad \int_x \sqrt{1 - c^2 (\sin x)^2}, \text{ \&c.}$$

II. To reduce a proposed integral to one or more of the elementary integrals.

This reduction, according to circumstances, is effected by some one of the following methods.

(1) By *substitution*, that is, by the introduction of a new variable.

(2) By altering its form by some common algebraical process, but without substitution.

(3) By the method of *rational fractions*, that is, by resolving *rational* expressions into the sums of several others of simpler forms.

(4) By *rationalization*, that is, by substituting in *irrational* expressions, so as to make them rational.

(5) By the application of *formulæ of reduction* whence a proposed integral is reduced to one more simple, and this

again to one yet more simple, and so on, till at last it is made to depend upon an elementary integral.

(6) By integration *by parts*, that is, by the employment of a certain general formula applicable to all cases; of this method, one or two of the above are only modifications.

5. The integrals of algebraic functions, as far as they can be obtained, are expressed either by algebraic expressions, or by Naperian logarithms, denoted $\log_\epsilon$, or by angles determined by their circular measure, viz. arc ÷ radius, or by Elliptic functions; for the numerical values of the three latter, when the values of the undetermined quantities which enter into them are assigned, recourse must be had to the proper Tables. It is indeed the existence of those Tables which has led to this mode of representing the values of integrals. The integrals of circular, logarithmic, and exponential functions will usually involve similar functions.

When a proposed integral cannot be obtained in a finite formula composed of any of the abovementioned quantities, it is expressed by an infinite series, (which is generally possible), so as to converge under the given circumstances.

6. Since any constant quantity connected with the variable part of an expression by the sign + or −, disappears in differentiating, it must be restored in integrating; and since all expressions which differ from one another only by their constant parts, have the same differential coefficient, we must in order to give an integral its most general form, (i. e. so as to comprehend all functions from which the proposed differential coefficient can have been derived), add to it an *indeterminate* constant which we shall denote by C. Although in finding integrals we shall usually omit the constant for the sake of conciseness, yet in all practical applications of the Integral Calculus, it must be invariably annexed, and its value then determined by the conditions of the problem.

If the value of the integral be known, corresponding to a particular value of x, then the constant may be determined; thus let

$$\int_x u = f(x) + C,$$

and let A be the value of the integral corresponding to $x = a$,

$$\therefore\ A = f(a) + C,$$

$$\text{or } C = A - f(a);$$

$$\therefore\ \int_x u = f(x) - f(a) + A.$$

If $A = 0$, that is, if the integral vanish when $x = a$, the equation becomes

$$\int_x u = f(x) - f(a);$$

in this case (a) is called the *origin* of the integral.

In those formulæ which retain the sign of integration $\int_x$, the constant is unnecessary, being reserved under that sign.

7. Since $d_x(au) = a\,d_x u$, $\therefore$ the integral of $a\,d_x u$, or

$$\int_x (a\,d_x u) = au;$$

that is, if a constant quantity multiply a differential coefficient as a factor, it will also multiply its integral. Also since

$$\int_x (a\,d_x u) = au = a\int_x (d_x u),$$

a constant factor which multiplies a differential coefficient, may be written without the sign of integration; and any constant factor may be introduced under the sign, provided we place its reciprocal without the sign. Hence

$$\int_x (-u) = -\int_x u,$$

-1, being the factor brought out.

8. Let y_1, y_2, y_3, be functions of x, and u_1, u_2, u_3, their differential coefficients, so that $d_x y_1 = u_1$, and $\therefore\ y_1 = \int_x u_1$, &c.;

$$\therefore\ d_x(y_1 + y_2 - y_3) = u_1 + u_2 - u_3;$$

$$\therefore\ \int_x (u_1 + u_2 - u_3) = y_1 + y_2 - y_3 = \int_x u_1 + \int_x u_2 - \int_x u_3,$$

that is, the integral of the sum or difference of several differential coefficients, is equal to the sum or difference of the integrals of these differential coefficients.

9. Since $d_x(UV)=Ud_xV+Vd_xU$, $\therefore\ UV=\int_x Ud_xV+\int_x Vd_xU$,

$$\text{or } \int_x Ud_xV = UV - \int_x Vd_xU.$$

This result shews that when a function can be resolved into two factors, the integral of one of which d_xV can be obtained, its integration depends upon the integration of the product of the integral V already found, and the differential coefficient d_xU of the unintegrated factor. The above is the fundamental formula for *integration by parts*, and is the one alluded to in the last of the methods of integration enumerated in Art. 4. If $V=x$, and $\therefore\ d_xV=1$, the formula becomes

$$\int_x U = Ux - \int_x x d_xU.$$

10. Since $d_x\left(\dfrac{U}{V}\right)=\dfrac{d_xU}{V}-\dfrac{Ud_xV}{V^2}$, $\therefore\ \dfrac{U}{V}=\displaystyle\int_x\frac{d_xU}{V}-\int_x\frac{Ud_xV}{V^2}$,

$$\text{or } \int_x \frac{d_xU}{V} = \frac{U}{V} + \int_x \frac{U}{V^2}\, d_xV.$$

11. Since $d_x x^m = mx^{m-1}$ for all values of m,

$$\therefore\ \int_x mx^{m-1} = x^m + C, \text{ or } m\int_x x^{m-1} = x^m + C;$$

$$\therefore\ \int_x x^{m-1} = \frac{x^m}{m} + C_1,$$

$\left(\text{putting the constant } C_1 \text{ instead of } \dfrac{C}{m}\right)$ that is, the integral of any power of x is found by *adding* 1 *to the index, and dividing by the increased index.*

If it be given that $\int_x x^{m-1}$ vanishes when $x=a$, then

$$0 = \frac{a^m}{m} + C_1.$$

which determines C_1; and subtracting this equation from the former to eliminate C_1, the corrected integral is

$$\int_x x^{m-1} = \frac{x^m - a^m}{m}.$$

Hence $\int_x \frac{1}{x^n} = -\frac{1}{n-1} \cdot \frac{1}{x^{n-1}}$, $\int_x x^{\frac{m}{n}} = \frac{n x^{\frac{m}{n}+1}}{m+n}$, $\int_x \frac{1}{\sqrt{x}} = 2\sqrt{x}$.

12. Generally, if u be a function of x, and m any number whatever,

$$\text{since } d_x a u^m = m a u^{m-1} d_x u,$$

$$\therefore \int_x m a u^{m-1} d_x u = a u^m, \text{ or } \int_x u^{m-1} a d_x u = \frac{a u^m}{m};$$

$$\text{now let } a d_x u = v, \therefore \int_x u^{m-1} v = \frac{a u^m}{m} = \frac{u^m v}{m d_x u}.$$

Hence if an expression be of the form $u^{m-1}v$, (the quantity which multiplies u^{m-1}, that is, which is without the vinculum, bearing a constant ratio to $d_x u$ the differential coefficient of the quantity under the vinculum) its integral is found by this Rule, *In the proposed expression add* 1 *to the index, divide by the increased index, and by the differential coefficient of the quantity under the vinculum.*

Ex. 1. $\int_x (a^m + x^m)^{n-1} x^{m-1}$.

The differential coefficient of the quantity under the vinculum, is $m x^{m-1}$, which bears a constant ratio to x^{m-1}, the quantity without the vinculum,

$$\therefore \int_x (a^m + x^m)^{n-1} x^{m-1} = \frac{(a^m + x^m)^n x^{m-1}}{n \cdot m x^{m-1}} = \frac{1}{mn}(a^m + x^m)^n.$$

Ex. 2. $\int_x (a + b x^n)^{\frac{p}{q}} x^{n-1} = \frac{(a + b x^n)^{\frac{p}{q}+1} x^{n-1}}{\left(\frac{p}{q} + 1\right) n b x^{n-1}}$

$$= \frac{q}{nb(p+q)}(a + b x^n)^{\frac{p}{q}+1}.$$

✓ Ex. 3. $\int_x (a^4 - x^4)^{\frac{5}{3}} 3x^3 = \dfrac{(a^4 - x^4)^{\frac{8}{3}} 3x^3}{\frac{8}{3}(-4x^3)} = -\dfrac{9}{32}(a^4 - x^4)^{\frac{8}{3}}.$

✓ Ex. 4. $\int_x \dfrac{a}{(x-b)^n} = \dfrac{a(x-b)^{-n+1}}{-n+1} = -\dfrac{a}{n-1} \cdot \dfrac{1}{(x-b)^{n-1}}.$

✓ Ex. 5. $\int_x u^{-\frac{1}{2}} d_x u = 2\sqrt{u}$, or $\int_x \dfrac{d_x u}{\sqrt{u}} = 2\sqrt{u}$;

that is, *if an expression be of the form of the differential coefficient of a function of* x, *divided by the square root of that function, its integral is equal to twice the square root of that function.* Hence

$$\int_x \frac{x}{\sqrt{a^2 + x^2}} = \frac{1}{2}\int_x \frac{2x}{\sqrt{a^2 + x^2}} = \frac{1}{2} \cdot 2\sqrt{a^2 + x^2} = \sqrt{a^2 + x^2};$$

also, $\int_x \dfrac{a - x}{\sqrt{2ax - x^2}} = \dfrac{1}{2}\int_x \dfrac{d_x (2ax - x^2)}{\sqrt{2ax - x^2}} = \sqrt{2ax - x^2}.$

✓ Ex. 6. $\int_x \dfrac{1}{x}(a + b\log_\epsilon x)^n = \dfrac{\frac{1}{x}(a + b\log_\epsilon x)^{n+1}}{(n+1)\frac{b}{x}}$

$$= \frac{(a + b\log_\epsilon x)^{n+1}}{(n+1)b}.$$

Ex. 7. $\int_x (\sin x)^n \cos x = \int_x (\sin x)^n \cdot d_x (\sin x) = \dfrac{(\sin x)^{n+1}}{n+1}.$

13. Expressions which do not appear under the proper form for the immediate application of the rule in the preceding article, may sometimes be reduced to it.

Ex. 1. $\int_x \dfrac{1}{(a^2 \pm x^2)^{\frac{3}{2}}};$

$\frac{1}{(a^2 \pm x^2)^{\frac{3}{2}}}$ is not of the proper form, but may be altered so as to become so, for it

$$= \frac{1}{x^3 (a^2 x^{-2} \pm 1)^{\frac{3}{2}}} = x^{-3} (a^2 x^{-2} \pm 1)^{-\frac{3}{2}},$$

which is of the proper form;

$$\therefore \int_x \frac{1}{(a^2 \pm x^2)^{\frac{3}{2}}} = \frac{x^{-3} (a^2 x^{-2} \pm 1)^{-\frac{1}{2}}}{-\frac{1}{2}(-2a^2 x^{-3})}$$

$$= \frac{(a^2 x^{-2} \pm 1)^{-\frac{1}{2}}}{a^2} = \frac{x}{a^2 \sqrt{a^2 \pm x^2}}.$$

Similarly, $\int_x \frac{1}{(a^n + x^n)^{\frac{n+1}{n}}} = \frac{x}{a^n (a^n + x^n)^{\frac{1}{n}}}.$

✓ **Ex. 2.** $\int_x \frac{1}{x\sqrt{2ax - x^2}} = \int_x \frac{1}{x^2 \sqrt{2ax^{-1} - 1}}$

$$= \int_x x^{-2} (2ax^{-1} - 1)^{-\frac{1}{2}} = \frac{x^{-2}(2ax^{-1} - 1)^{\frac{1}{2}}}{\frac{1}{2}(-2ax^{-2})} = -\frac{\sqrt{2ax^{-1} - 1}}{a},$$

$$\text{or } \int_x \frac{1}{x\sqrt{2ax - x^2}} = -\frac{\sqrt{2ax - x^2}}{ax}.$$

+

Ex. 3. $\int_x \{(\tan x)^{n-1} + (\tan x)^{n+1}\} =$

$$\int_x \{1 + (\tan x)^2\} (\tan x)^{n-1} = \int_x (\tan x)^{n-1} d_x \tan x = \frac{1}{n} (\tan x)^n.$$

✓ **Ex. 4.** $\int_x \frac{1}{\sqrt{e^{2x} + e^x}} = \int_x \frac{1}{e^x \sqrt{1 + e^{-x}}}$

$$= \int_x e^{-x} (1 + e^{-x})^{-\frac{1}{2}} = -\int_x d_x e^{-x} (1 + e^{-x})^{-\frac{1}{2}} = -2\sqrt{1 + e^{-x}}.$$

✓ **14.** Since $d_x \log_e x = \frac{1}{x}$,

$$\therefore \int_x \frac{1}{x} = \log_e x + C.$$

If in the equation $\int_x x^m = \frac{x^{m+1}}{m+1}$, (Art. 11.) we make $m = -1$, we find $\int_x \frac{1}{x} = \infty$; hence the rule for finding $\int_x x^m$ is said to fail when $m = -1$; the reason of this discrepancy is, that the equation $\int_x \frac{1}{x} = \log_e x$, supposes the function of x denoted by $\int_x \frac{1}{x}$ to vanish when $x = 1$, whilst the equation $\int_x x^m = \frac{x^{m+1}}{m+1}$ supposes $\int_x x^m$ to vanish when $x = 0$; if however we introduce the same supposition into the latter equation as is made in the former, we shall find accordant results; for in that case, by introducing the constant, we have,

$$\int_x x^m = \frac{x^{m+1} - 1}{m+1},$$

and if $m = -1$, the second member assumes the form $\frac{0}{0}$; to obtain the true value, let $m = -1 + h$, h being very small, so that for x^h we may use its expansion in powers of h;

$$\therefore \frac{x^{m+1} - 1}{m+1} = \frac{x^h - 1}{h} = \frac{1}{h}\left(1 + h \log_e x + \frac{h^2 (\log_e x)^2}{1 \,.\, 2} + \&c. - 1\right)$$

$$= \log_e x + \frac{h}{2} (\log_e x)^2 + \&c.$$

Now let $h = 0$, or $m = -1$, $\therefore \int_x \frac{1}{x} = \log_e x$.

15. Generally, since $d_x (\log_e u) = \frac{d_x u}{u}$,

$$\therefore \int_x \frac{d_x u}{u} = \log_e u + C, \text{ and } \int_x \frac{m d_x u}{u} = m \int_x \frac{d_x u}{u}$$

$$= m \log_e u + C = \log_e \left(\frac{u}{c}\right)^m, \text{ making } C = -m \log_e c.$$

Hence *the integral of any fraction whose numerator is the differential coefficient of the denominator, or bears a constant ratio to it, is the Naperian logarithm of the denominator, or bears that same ratio to it.*

✓ Ex. 1. $$\int_x \frac{b}{x \pm a} = b \int_x \frac{d_x(x \pm a)}{x \pm a} = b \log_\epsilon (x \pm a).$$

✓ Ex. 2. $$\int_x \frac{a}{a + bx} = \frac{a}{b} \int_x \frac{b}{a + bx} = \frac{a}{b} \int_x \frac{d_x(a + bx)}{a + bx}$$

$$= \frac{a}{b} \log_\epsilon (a + bx).$$

✓ Ex. 3. $$\int_x \frac{x^{n-1}}{a^n + x^n} = \frac{1}{n} \int_x \frac{nx^{n-1}}{a^n + x^n}$$

$$= \frac{1}{n} \log_\epsilon (a^n + x^n) = \log_\epsilon (a^n + x^n)^{\frac{1}{n}}.$$

Ex. 4. $$\int_x \frac{1}{x(a + bx)} = \int_x \frac{1}{x^2 (ax^{-1} + b)}$$

$$= -\frac{1}{a} \int_x \frac{-ax^{-2}}{ax^{-1} + b} = -\frac{1}{a} \log_\epsilon (ax^{-1} + b) = \frac{1}{a} \log_\epsilon \left(\frac{x}{a + bx}\right).$$

16. Hence we are enabled to integrate all rational integral algebraic functions of x; for the most general form of such functions is $ax^m + bx^n + cx^r + \ldots + e$; the integral of which is

$$\frac{ax^{m+1}}{m + 1} + \frac{bx^{n+1}}{n + 1} + \ldots + ex + C.$$

It is not necessary that m, n, r, &c. should be positive integers; they may be negative or fractional, and the integral will be the same, except in the case in which any of the terms are of the form $\frac{g}{x}$, the integral of which is $g \log_\epsilon x$.

We shall proceed to show how several expressions, by a simple transformation, may be reduced to the above form; but must previously establish a formula for changing the hypothesis of integration, corresponding to the one in the Differential Calculus for changing the independant variable.

17. To prove that $\int_x \{f(z)\,d_x z\} = \int_z f(z)$, where z denotes a function of x, and $f(z)$ a function of z.

$$\text{Let } y = \int_x \{f(z)\,d_x z\}, \quad \therefore\ d_x y = f(z)\,d_x z,$$

$$\therefore\ \frac{d_x y}{d_x z} = f(z), \text{ or } d_z y = f(z), \quad \therefore\ y = \int_z f(z).$$

Hence, equating these two values of y,

$$\int_x \{f(z)\,d_x z\} = \int_z f(z).$$

This is a formula of great use; for in finding $\int_x u$, one method of most extensive application is that of *substitution*, that is, assuming $x = \phi(z)$ or some relation between x and z so as to obtain a value of u in the form $f(z)\,d_x z$, where $f(z)$ is a rational function of z, or an expression more easy to integrate than u, then

$$\int_x u = \int_x \{f(z)\,d_x z\} = \int_z f(z).$$

Of this method we shall see many examples in the following articles.

18. The formula of the preceding article may also be put under the form

$$\int_x u = \int_z u\,d_z x,$$

u being a function of x, and x a function of z;

$$\text{for let } y = \int_x u; \quad \therefore\ d_x y = u; \quad \therefore\ \frac{d_z y}{d_z x} = u;$$

$$\text{or } d_z y = u\,d_z x; \quad \therefore\ y = \int_z u\,d_z x; \quad \therefore\ \int_x u = \int_z u\,d_z x.$$

Hence in cases where it is more convenient to obtain $u\,d_z x$ in the form $f(z)$, than to obtain u in the form $f(z)\,d_x z$, we may prepare the integral $\int_x u$ for substitution, by writing it $\int_z u\,d_z x$.

It appears then that when we have an expression to be integrated with respect to x, *by multiplying it by* $d_z x$, we may transform the integral into one with respect to z.

19. All integrals falling under the form

$$\int_x (a + bx^n)^{\frac{p}{q}} x^{m-1},$$

can be reduced to the sum of a series of terms of the form $c\int_z z^r$, if $\frac{m}{n}$ = a positive integer, or $\frac{m}{n} + \frac{p}{q}$ = a negative integer.

Assume $a + bx^n = z$; $\therefore\ x^n = \frac{z-a}{b}$, and $x^m = \frac{(z-a)^{\frac{m}{n}}}{b^{\frac{m}{n}}}$;

$$\therefore\ x^{m-1} = \frac{1}{nb^{\frac{m}{n}}}(z-a)^{\frac{m}{n}-1} d_x z;$$

$$\therefore\ x^{m-1}(a+bx^n)^{\frac{p}{q}} = \frac{z^{\frac{p}{q}}}{nb^{\frac{m}{n}}}(z-a)^{\frac{m}{n}-1} d_x z; \quad \therefore\ \int_x x^{m-1}(a+bx^n)^{\frac{p}{q}}$$

$$= \frac{1}{nb^{\frac{m}{n}}}\int_x z^{\frac{p}{q}}(z-a)^{\frac{m}{n}-1} d_x z = \frac{1}{nb^{\frac{m}{n}}}\int_z z^{\frac{p}{q}}(z-a)^{\frac{m}{n}-1}, \text{ by Art. (17).}$$

Now if $\frac{m}{n}$ = a positive integer, $(z-a)^{\frac{m}{n}-1}$ may be expanded in a limited number of terms by the Binomial Theorem; and each term being multiplied by $z^{\frac{p}{q}}$, may be integrated by the rule of Art. (11).

Again, $(a+bx^n)^{\frac{p}{q}} x^{m-1} = (ax^{-n}+b)^{\frac{p}{q}} x^{m+\frac{np}{q}-1}$; and this latter expression, by what has just been said, is integrable, if $m + \frac{np}{q} \div (-n)$ = a positive integer, and therefore $\frac{m}{n} + \frac{p}{q}$ = a negative integer.

20. Hence an expression of the form $(a + bx^n)^{\frac{p}{q}} x^{m-1}$ being proposed for integration, first ascertain *if the index of x without the vinculum, increased by 1, be a multiple of the index of x under the vinculum*, in which case, assume $a + bx^n = z$. *If not, add $\frac{p}{q}$, and if the result be a negative integer*, assume $ax^{-n} + b = z$, taking care to put the expression under the form

$$(ax^{-n} + b)^{\frac{p}{q}} x^{m + \frac{np}{q} - 1}.$$

The assumption $ax^{-n} + b = z^q$, gives

$$\int_x (a + bx^n)^{\frac{p}{q}} x^{m-1} = -\frac{q}{na} \int_z z^{p+q-1} \left(\frac{z^q - b}{a}\right)^{-(\frac{m}{n} + \frac{p}{q} + 1)}.$$

Hence, if $\frac{m}{n} + \frac{p}{q} =$ a positive integer, the expression is transformed into a rational fraction, and therefore may be integrated as will be shewn; but this is not an advantageous practical method.

Ex. 1. $\int_x x^2 \sqrt{a + x}$,

here $\frac{2 + 1}{1} = 3$ a positive integer;

let $a + x = z$; $\therefore\ x = z - a$, and $x^3 = (z - a)^3$;

$$\therefore\ x^2 = (z - a)^2 d_x z;$$

$$\therefore\ x^2 \sqrt{a + x} = z^{\frac{1}{2}} (z - a)^2 d_x z = (z^{\frac{5}{2}} - 2az^{\frac{3}{2}} + a^2 z^{\frac{1}{2}}) d_x z;$$

$$\therefore\ \int_x x^2 \sqrt{a + x} = \int_z (z^{\frac{5}{2}} - 2az^{\frac{3}{2}} + a^2 z^{\frac{1}{2}}) = \frac{2}{7} z^{\frac{7}{2}} - \frac{4}{5} a z^{\frac{5}{2}} + \frac{2}{3} a^2 z^{\frac{3}{2}}$$

$$= \frac{2}{35} z^{\frac{3}{2}} (5z^2 - 14az + \frac{35}{3} a^2)$$

$$= \frac{2}{35} (a + x)^{\frac{3}{2}} \left(5x^2 - 4ax + \frac{8a^2}{3}\right).$$

Ex. 2. $\int_x \frac{x^{3n-1}}{\sqrt{a+bx^n}} = \frac{\sqrt{a+bx^n}}{15nb^3}(6x^{2n}b^2 - 8abx^n + 16a^2)$.

In this example, $\frac{(3n-1)+1}{n} = 3$.

Ex. 3. $\int_x \frac{x^m}{(a+bx)^n} = \frac{1}{b^{m+1}} \int_z \frac{(z-a)^m}{z^n}$ (making $a+bx = z$) which can be expanded and integrated when m is a positive integer.

Other expressions of frequent occurrence, which may be integrated by this rule, are $\frac{x^{2r-1}}{\sqrt{a^2 \mp x^2}}$, $\frac{x^m}{a+bx}$.

Ex. 4. $\int_x \frac{\sqrt[3]{1-x^3}}{x^5}$, here $\frac{-5+1}{3} = -\frac{4}{3}$ is not a positive integer; add $\frac{1}{3}$ and it becomes -1, this example then falls under the 2nd Case. Write it therefore, $\int_x (x^{-3}-1)^{\frac{1}{3}} x^{-4}$,

which (Art. 12.) $= \frac{(x^{-3}-1)^{\frac{4}{3}} x^{-4}}{\frac{4}{3}(-3x^{-4})} = -\frac{(1-x^3)^{\frac{4}{3}}}{4x^4}$.

Ex. 5. $\int_x \frac{1}{x^m(a+bx)^n} = \int_x x^{-(m+n)}(ax^{-1}+b)^{-n}$

$$= -\frac{1}{b^{m+n-1}} \int_z \frac{(z-a)^{m+n-2}}{z^n},$$

making $ax^{-1}+b = z$, which can be expanded and integrated, when $m+n-2$ is a positive integer.

Ex. 6. $\int_x \frac{1}{x^2(a^3+x^3)^{\frac{5}{3}}} = \int_x x^{-2}(a^3+x^3)^{-\frac{5}{3}}$.

Here $\frac{-2+1}{3} - \frac{5}{3} = -2$ a negative integer, write the expression therefore $x^{-7}(a^3x^{-3}+1)^{-\frac{5}{3}}$, which can be readily integrated without substitution; for it

$$=\frac{x^{-4}}{a^3}(a^3x^{-3}+1-1)(a^3x^{-3}+1)^{-\frac{5}{3}}$$

$$=\frac{1}{a^3}x^{-4}(a^3x^{-3}+1)^{-\frac{2}{3}}-\frac{1}{a^3}x^{-4}(a^3x^{-3}+1)^{-\frac{5}{3}},$$

the integral of which, by Art. 12., is

$$=-\frac{1}{a^6}\left\{\frac{(a^3+x^3)^{\frac{1}{3}}}{x}+\frac{1}{2}\,\frac{x^2}{(a^3+x^3)^{\frac{2}{3}}}\right\}.$$

Other expressions falling under this rule are

$$\int_x\frac{1}{(a^n+x^n)^{r+\frac{1}{n}}},\quad\int_x\frac{1}{x^{2r}\sqrt{a^2\mp x^2}}.$$

21. Integration of the elementary *logarithmic forms*.

These are certain expressions which do not appear under the fundamental form $\frac{m\,d_xu}{u}$, but may be made to assume it.

I. $$\int_x\frac{d_xu}{\sqrt{u^2\pm a^2}}.$$

Let $u^2\pm a^2=z^2$; $\therefore u\,d_xu=z\,d_xz$, or $\frac{d_xz}{d_xu}=\frac{u}{z}$,

$$\therefore\frac{d_xz+d_xu}{d_xu}=\frac{u+z}{z},\quad\text{or,}\quad\frac{d_xu}{z}=\frac{d_xz+d_xu}{z+u},$$

$$\therefore\frac{d_xu}{\sqrt{u^2\pm a^2}}=\frac{d_x(z+u)}{z+u};\ \text{hence (Art. 15.)}$$

$$\int_x\frac{d_xu}{\sqrt{u^2\pm a^2}}=\log.(u+z)=\log.(u+\sqrt{u^2\pm a^2})+C.$$

Suppose we have given that the value of the integral $\int_x \frac{d_x u}{\sqrt{u^2 - a^2}}$, corresponding to $u=a$, is 0,

$$\therefore\ 0 = \log_\epsilon (a) + C, \quad \text{or } C = -\log_\epsilon a.$$

$$\therefore\ \int_x \frac{d_x u}{\sqrt{u^2 - a^2}} = \log_\epsilon (u + \sqrt{u^2 - a^2}) - \log_\epsilon a,$$

$$= \log_\epsilon \left\{ \frac{u}{a} + \sqrt{\left(\frac{u}{a}\right)^2 - 1} \right\}.$$

II. $$\int_x \frac{d_x u}{u\sqrt{a^2 \pm u^2}}.$$

Let $u = \frac{a}{z}$, $\quad \therefore\ \log_\epsilon u = \log_\epsilon a - \log_\epsilon z, \quad \therefore\ \frac{1}{u} d_x u = -\frac{1}{z} d_x z,$

$$\therefore\ \int_x \frac{d_x u}{u\sqrt{a^2 \pm u^2}} = \int_x \frac{-d_x z}{z\sqrt{a^2 \pm a^2 z^{-2}}} = -\frac{1}{a} \int_x \frac{d_x z}{\sqrt{z^2 \pm 1}}.$$

$= -\frac{1}{a} \log_\epsilon (z + \sqrt{z^2 \pm 1})$, by the preceding case,

$$\text{or } \int_x \frac{d_x u}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a} \log_\epsilon \left(\frac{a + \sqrt{a^2 \pm u^2}}{u} \right)$$

$$= \frac{1}{a} \log_\epsilon \left(\frac{u}{a + \sqrt{a^2 \pm u^2}} \right) + C.$$

III. $$\int_x \frac{d_x u}{a^2 - u^2}.$$

Since $\frac{2a}{a^2 - u^2} = \frac{2a}{(a+u)(a-u)} = \frac{1}{a+u} + \frac{1}{a-u}$,

$$\therefore\ \frac{d_x u}{a^2 - u^2} = \frac{1}{2a} \left(\frac{d_x u}{a+u} - \frac{-d_x u}{a-u} \right);$$

$$\therefore \int_x \frac{d_x u}{a^2 - u^2} = \frac{1}{2a}\left\{\log_\epsilon (a+u) - \log_\epsilon (a-u)\right\}$$

$$= \frac{1}{2a} \log_\epsilon \left(\frac{a+u}{a-u}\right) + C.$$

IV. $$\int_x \frac{d_x u}{u^2 - a^2}.$$

Here $\dfrac{2a}{u^2 - a^2} = \dfrac{1}{u-a} - \dfrac{1}{u+a}$,

$$\therefore \int_x \frac{d_x u}{u^2 - a^2} = \frac{1}{2a}\left\{\log_\epsilon (u-a) - \log_\epsilon (u+a)\right\},$$

$$= \frac{1}{2a} \log_\epsilon \left(\frac{u-a}{u+a}\right) + C.$$

This integral may also be derived from the preceding one, by changing the constant into $-\dfrac{1}{2a}\log_\epsilon(-1) - C$.

22. Integration of elementary *circular forms.*

By reversing the rules for finding the differential coefficients of the inverse circular functions, we obtain the following results:

I. Since $d_x \sin^{-1}\dfrac{u}{a} = \dfrac{d_x \dfrac{u}{a}}{\sqrt{1 - \left(\dfrac{u}{a}\right)^2}} = \dfrac{d_x u}{\sqrt{a^2 - u^2}}$,

$$\therefore \int_x \frac{d_x u}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C.$$

Also, since $\sin^{-1}\dfrac{u}{a} = \dfrac{\pi}{2} - \cos^{-1}\dfrac{u}{a}$, we have

$$\int_x \frac{-d_x u}{\sqrt{a^2 - u^2}} = \cos^{-1}\frac{u}{a} + C,$$

(including $\frac{\pi}{2}$ in the constant, and changing the signs,) at which we might arrive immediately by observing that

$$d_x \cos^{-1}\frac{u}{a} = \frac{-d_x u}{\sqrt{a^2 - u^2}}.$$

II. Since $d_x \sec^{-1}\frac{u}{a} = \dfrac{d_x \frac{u}{a}}{\frac{u}{a}\sqrt{\left(\frac{u}{a}\right)^2 - 1}} = a\dfrac{d_x u}{u\sqrt{u^2 - a^2}}$,

$$\therefore \int_x \frac{d_x u}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C.$$

III. Since $d_x \tan^{-1}\frac{u}{a} = \dfrac{d_x \frac{u}{a}}{1 + \left(\frac{u}{a}\right)^2} = a\dfrac{d_x u}{u^2 + a^2}$,

$$\therefore \int_x \frac{d_x u}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C.$$

IV. Since $d_x \operatorname{versin}^{-1}\frac{u}{a} = \dfrac{d_x \frac{u}{a}}{\sqrt{2\frac{u}{a} - \left(\frac{u}{a}\right)^2}} = \dfrac{d_x u}{\sqrt{2au - u^2}}$,

$$\therefore \int_x \frac{d_x u}{\sqrt{2au - u^2}} = \operatorname{versin}^{-1}\frac{u}{a} + C.$$

These four formulæ, which are called *circular* forms, together with the four *logarithmic* forms investigated in the preceding article, must be carefully recollected. Joined with the expressions

$$\int_x u^m d_x u = \frac{u^{m+1}}{m+1}, \quad \text{and} \quad \int_x \frac{m d_x u}{u} = m \log_\epsilon u,$$

they constitute fundamental formulæ, to one or more of which it is the object of almost every process in the Integral Calculus to reduce the integrals of proposed algebraical expressions.

23. The following integrals are also of very frequent occurrence; they furnish examples of integration by parts, Art. 9, being by that means reduced to the preceding forms.

I. $$\int_x \sqrt{u^2+a^2}\, d_x u.$$

In the formula $\int_x U d_x V = UV - \int_x V d_x U$,

make $U = \sqrt{u^2+a^2}$, $\therefore\ d_x U = \dfrac{u d_x u}{\sqrt{u^2+a^2}}$, and $V = u$,

$$\therefore\ \int_x \sqrt{u^2+a^2}\, d_x u = u\sqrt{u^2+a^2} - \int_x u \frac{u d_x u}{\sqrt{u^2+a^2}}$$

$$= u\sqrt{u^2+a^2} - \int_x \frac{(u^2+a^2-a^2)}{\sqrt{u^2+a^2}}\, d_x u$$

$$= u\sqrt{u^2+a^2} - \int_x \sqrt{u^2+a^2}\, d_x u + a^2 \int_x \frac{d_x u}{\sqrt{u^2+a^2}},$$

$$\therefore\ 2\int_x \sqrt{u^2+a^2}\, d_x u = u\sqrt{u^2+a^2} + a^2 \log_\epsilon (u + \sqrt{u^2+a^2}),$$

by first logarithmic form, Art. 21, or

$$\int_x \sqrt{u^2+a^2}\, d_x u = \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\log_\epsilon (u + \sqrt{u^2+a^2}).$$

Similarly,

$$\int_x \sqrt{u^2-a^2}\, d_x u = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\log_\epsilon (u + \sqrt{u^2-a^2}),$$

writing $-a^2$, instead of a^2.

II. $$\int_x \sqrt{a^2 - u^2}\, d_x u.$$

Integrating by parts,

$$\int_x \sqrt{a^2 - u^2}\, d_x u = u\sqrt{a^2 - u^2} - \int_x u \frac{-u\, d_x u}{\sqrt{a^2 - u^2}}$$

$$= u\sqrt{a^2 - u^2} - \int_x \frac{a^2 - u^2 - a^2}{\sqrt{a^2 - u^2}}\, d_x u$$

$$= u\sqrt{a^2 - u^2} - \int_x \sqrt{a^2 - u^2}\, d_x u + a^2 \int_x \frac{d_x u}{\sqrt{a^2 - u^2}},$$

$$\therefore \int_x \sqrt{a^2 - u^2}\, d_x u = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a},$$

by first circular form, Art. 22.

III. $$\int_x \sqrt{u^2 + 2au}\, d_x u$$

$$= \int_x \sqrt{(u + a)^2 - a^2}\, d_x (u + a)$$

$$= \frac{u + a}{2}\sqrt{u^2 + 2au} - \frac{a^2}{2} \log_e (u + a + \sqrt{u^2 + 2au}),$$

by formula I, a being either positive or negative.

IV. $$\int_x \sqrt{2au - u^2}\, d_x u$$

$$= \int_x \sqrt{a^2 - (u - a)^2}\, d_x (u - a)$$

$$= \frac{u - a}{2}\sqrt{2au - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u - a}{a},$$ by formula II.

24. Although we have given separate investigations of the circular and logarithmic forms, it is easy to deduce one from the other. Take for instance the form

$$\int_x \frac{1}{\sqrt{x^2 - a^2}} = \log_e (x + \sqrt{x^2 - a^2}) + C,$$ (Art. 21. making $u = x$),

and in order that the integral may vanish when $x = a$, make $C = -\log_\epsilon a$,

$$\therefore\ \log_\epsilon\left\{\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 - 1}\right\} = \int_x \frac{1}{\sqrt{x^2 - a^2}} = \sqrt{-1}\int_x \frac{-1}{\sqrt{a^2 - x^2}}$$

$$\left.\begin{aligned} &= z\sqrt{-1},\ \text{suppose; therefore } \frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 - 1} = e^{z\sqrt{-1}} \\ &\text{and } \left\{\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 - 1}\right\}^{-1} = \frac{x}{a} - \sqrt{\left(\frac{x}{a}\right)^2 - 1} = e^{-z\sqrt{-1}} \end{aligned}\right\},$$

$$\therefore\ \frac{x}{a} = \tfrac{1}{2}\left(e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}\right) = \cos z,$$

$$\therefore\ z = \cos^{-1}\frac{x}{a},\quad \text{or} \int_x \frac{-1}{\sqrt{a^2 - x^2}} = \cos^{-1}\frac{x}{a}.$$

25. We shall now proceed to shew the use of the circular and logarithmic forms in integrating a variety of expressions, which are either of the proper form for the immediate application of them, or can be reduced to that form.

If in Articles 21, 22, and 23, we suppose the function of x denoted by u, to be $= x$, and therefore $d_x u = 1$, we obtain the following results:

✓ $$\int_x \frac{1}{\sqrt{x^2 \pm a^2}} = \log_\epsilon\left(x + \sqrt{x^2 \pm a^2}\right).$$

✓ $$\int_x \frac{1}{x\sqrt{a^2 \pm x^2}} = \frac{1}{a}\log_\epsilon\left(\frac{x}{a + \sqrt{a^2 \pm x^2}}\right).$$

✓ $$\int_x \frac{1}{a^2 - x^2} = \frac{1}{2a}\log_\epsilon\left(\frac{a + x}{a - x}\right).$$

✓ $$\int_x \frac{1}{x^2 - a^2} = \frac{1}{2a}\log_\epsilon\left(\frac{x - a}{x + a}\right).$$

$$\int_x \frac{1}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}.$$

$$\int_x \frac{1}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}.$$

$$\int_x \frac{1}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}.$$

$$\int_x \frac{1}{\sqrt{2ax - x^2}} = \text{versin}^{-1}\frac{x}{a}.$$

$$\int_x \sqrt{x^2 \pm a^2} = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\log_\epsilon\left(x + \sqrt{x^2 \pm a^2}\right).$$

$$\int_x \sqrt{a^2 - x^2} = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

$$\int_x \sqrt{x^2 + 2ax} = \frac{x + a}{2}\sqrt{x^2 + 2ax} - \frac{a^2}{2}\log_\epsilon\left(x + a + \sqrt{x^2 + 2ax}\right).$$

$$\int_x \sqrt{2ax - x^2} = \frac{x - a}{2}\sqrt{2ax - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x - a}{a}.$$

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It may be useful to observe that each of the above results is homogeneous in x and a, and of a dimension greater by unity than the differential coefficient (also homogeneous in x and a) of which it is the integral.

26. In the following examples, the quantity to be integrated requires a previous reduction to the proper form.

1. $$\int_x \frac{1}{\sqrt{x^2 + 2ax}} = \int_x \frac{d_x(x + a)}{\sqrt{(x + a)^2 - a^2}}$$

$= \log_\epsilon\left(x + a + \sqrt{x^2 + 2ax}\right)$, by first logarithmic form.

2. $$\int_x \frac{1}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{c}}\int_x \frac{2c}{\sqrt{4c(a + bx + cx^2)}}$$

$$= \frac{1}{\sqrt{c}} \int_x \frac{d_x(2cx+b)}{\sqrt{(2cx+b)^2 + 4ac - b^2}}$$

$$= \frac{1}{\sqrt{c}} \log_e \left\{2cx + b + \sqrt{4c\,(a + bx + cx^2)}\right\}.$$

3. $$\int_x \frac{1}{a+bx^2} = \int_x \frac{b}{ab + b^2x^2} = \int_x \frac{d_x(bx)}{ab+b^2x^2}$$

$$= \frac{1}{\sqrt{ab}} \tan^{-1} x \sqrt{\frac{b}{a}}.$$

4. $$\int_x \frac{1}{a+bx+cx^2} = 2\int_x \frac{2c}{4ac + 4bcx + 4c^2x^2}$$

$$= 2\int_x \frac{d_x(2cx+b)}{(2cx+b)^2 + 4ac - b^2} = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \frac{2cx+b}{\sqrt{4ac-b^2}},$$

if $4ac > b^2$, by third circular form, Art. 22.

Since $\tan^{-1} x = \sec^{-1}\sqrt{1+x^2} = \cot^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{x}$

$$= \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \operatorname{versin}^{-1}\left(1 - \frac{1}{\sqrt{1+x^2}}\right);$$

making $\sqrt{4ac - b^2} = k$, the expression $\frac{2}{k}\tan^{-1}\frac{2cx+b}{k}$ may be replaced by

$$\frac{2}{k}\sec^{-1}\sqrt{1 + \frac{(2cx+b)^2}{k^2}} = \frac{2}{k}\sec^{-1}\frac{2\sqrt{c}}{k}\sqrt{a+bx+cx^2};$$

similarly, substituting $\frac{2cx+b}{k}$ for x in any other of the above formulæ, we may obtain different expressions for the integral, and it is obvious that every integral expressed by an angle will admit of similar transformations.

If $4ac < b^2$, the value obtained above for the integral becomes imaginary; in that case, by the fourth logarithmic form, Art. 21, we have

$$\int_x \frac{1}{a+bx+cx^2} = 2\int_x \frac{d_x(2cx+b)}{(2cx+b)^2-(b^2-4ac)}$$

$$= \frac{1}{\sqrt{b^2-4ac}}\cdot\log_e \frac{2cx+b-\sqrt{b^2-4ac}}{2cx+b+\sqrt{b^2-4ac}}.$$

5. $$\int_x \frac{x}{\sqrt{a^4-x^4}} = \tfrac{1}{2}\int_x \frac{d_x x^2}{\sqrt{a^4-(x^2)^2}} = \tfrac{1}{2}\sin^{-1}\frac{x^2}{a^2}.$$

6. $$\int_x \frac{1}{\sqrt{a+bx-cx^2}} = \frac{1}{\sqrt{c}}\int_x \frac{2c}{\sqrt{4c(a+bx-cx^2)}}$$

$$= \frac{1}{\sqrt{c}}\int_x \frac{d_x(2cx-b)}{\sqrt{4ac+b^2-(2cx-b)^2}} = \frac{1}{\sqrt{c}}\sin^{-1}\frac{2cx-b}{\sqrt{4ac+b^2}},$$

by first circular form, Art. 22.

7. $$\int_x \frac{x}{\sqrt{(x^2-a^2)(b^2-x^2)}} = \int_x \frac{d_x\sqrt{x^2-a^2}}{\sqrt{b^2-a^2-(x^2-a^2)}}$$

$$= \sin^{-1}\sqrt{\frac{x^2-a^2}{b^2-a^2}}.$$

27. In some cases it is more convenient to reduce $\int_x u$ to the proper form, by substitution.

Ex. 1. $\int_x \frac{1}{x\sqrt{a+bx^n}}$; let $bx^n = z^2$, $\therefore \log_e b + n\log_e x = 2\log_e z$,

$$\therefore \text{ by differentiation } \frac{n}{x} = \frac{2d_x z}{z};$$

$$\therefore\ \int_x u = \frac{2}{n}\int_z \frac{d_z z}{z\sqrt{a+z^2}}$$

$$= \frac{2}{n\sqrt{a}}\log_\epsilon \frac{z}{\sqrt{a}+\sqrt{a+z^2}} = \frac{2}{n\sqrt{a}}\log_\epsilon\left(\frac{\sqrt{bx^n}}{\sqrt{a}+\sqrt{a+bx^n}}\right).$$

2. $\displaystyle\int_x \frac{1}{x\sqrt{ax^{-n}-b}}$; let $ax^{-n} = z^2$, $\displaystyle\therefore\ \frac{1}{x} = -\frac{2}{n}\,\frac{d_x z}{z}$;

$$\therefore\ \int_x u = -\frac{2}{n}\int_z \frac{d_z z}{z\sqrt{z^2-b}}$$

$$= -\frac{2}{n\sqrt{b}}\sec^{-1}\frac{z}{\sqrt{b}} = -\frac{2}{n\sqrt{b}}\sec^{-1}\sqrt{\frac{a}{bx^n}}.$$

3. $\displaystyle\int_x \frac{1}{x\sqrt{a+x}} = \frac{2}{\sqrt{a}}\log_\epsilon\left(\frac{\sqrt{x}}{\sqrt{a}+\sqrt{a+x}}\right).$

4. $\displaystyle\int_x \frac{1}{x\sqrt{a+bx+cx^2}}$; let $x = \dfrac{1}{z}$, $\displaystyle\therefore\ \frac{1}{x} = -\frac{d_x z}{z}$;

$$\therefore\ \int_x u = -\int_z \frac{d_z z}{z\sqrt{a+\dfrac{b}{z}+\dfrac{c}{z^2}}} = -\int_z \frac{1}{\sqrt{az^2+bz+c}}$$

$$= -\frac{1}{\sqrt{a}}\log_\epsilon\left\{2az+b+\sqrt{4a\,(az^2+bz+c)}\right\},\ \text{(Ex. 2. Art. 26.)}$$

$$= -\frac{1}{\sqrt{a}}\log_\epsilon\left\{\frac{2a}{x}+b+\sqrt{4a\left(\frac{a}{x^2}+\frac{b}{x}+c\right)}\right\}.$$

5. $\displaystyle\int_x \frac{1}{(c+ex)\sqrt{a+bx}}$; let $\sqrt{a+bx} = z$,

$$\therefore\ \frac{b}{2\sqrt{a+bx}} = d_x z,\ \text{and}\ c+ex = \frac{bc+e\,(z^2-a)}{b};$$

$$\therefore\ \int_z u = 2\int_z \frac{d_z z}{bc - ea + ez^2} = 2\int_z \frac{d_z e z}{e(bc-ae) + (ez)^2}$$

$$= \frac{2}{\sqrt{e(bc-ae)}}\tan^{-1}\frac{ez}{\sqrt{e(bc-ae)}} = \frac{2}{\sqrt{e(bc-ae)}}\tan^{-1}\sqrt{\frac{e(a+bx)}{bc-ae}};$$

if $bc < ae$, the expression falls under the fourth logarithmic form, Art. 21.

6. $\displaystyle\int_x \frac{1}{(c+ex)\sqrt{a+bx^2}}$; let $c + ex = z$, $\therefore\ \dfrac{e}{c+ex} = \dfrac{d_x z}{z}$, //

$$\text{and } a + bx^2 = a + \frac{b}{e^2}(z-c)^2;$$

$$\therefore\ \int_x u = \int_z \frac{d_z z}{z\sqrt{ae^2 + b(z-c)^2}} = \int_z \frac{1}{z\sqrt{ae^2+bc^2-2bcz+bz^2}}$$

which was integrated Ex. 4.

7. $\displaystyle\int_x \frac{1}{(c+ex^2)\sqrt{a+bx^2}} = \int_x \frac{x^{-3}}{(cx^{-2}+e)\sqrt{ax^{-2}+b}}$ //

$$= -\frac{1}{a}\int_x \frac{d_x\sqrt{ax^{-2}+b}}{cx^{-2}+e};$$

let $\sqrt{ax^{-2}+b} = z$, $\therefore\ acx^{-2} + ae = cz^2 + ae - cb$,

$$\therefore\ \int_x u = -\int_z \frac{d_z z}{cax^{-2}+ae} = -\int_z \frac{1}{cz^2+ae-cb}$$

$$= -\frac{1}{\sqrt{c(ae-cb)}}\tan^{-1}\frac{cz}{\sqrt{c(ae-cb)}}$$

$$= -\frac{1}{\sqrt{c(ae-cb)}}\tan^{-1}\sqrt{\frac{c(a+bx^2)}{x^2(ae-cb)}}.$$

If $cb > ae$, the integral falls under a logarithmic form; and if $cb = ae$,

$$\int_x u = -\int_z \frac{1}{cz^2} = \frac{1}{cz} = \frac{x}{c\sqrt{a+bx^2}}.$$

8. $$\int_x \frac{x}{(c+ex^2)\sqrt{a+bx^2}} = \frac{1}{b}\int_x \frac{d_x\sqrt{a+bx^2}}{c+ex^2}$$

$$= \int_z \frac{1}{bc+e(z^2-a)}, \text{ making } \sqrt{a+bx^2} = z,$$

which is a known logarithmic or circular form, according as $bc <$ or $> ae$.

9. $$\int_x \frac{1-x^2}{1+x^2}\,\frac{1}{\sqrt{1+ax^2+x^4}} = \int_x \frac{x^{-2}-1}{x^{-1}+x}\,\frac{1}{\sqrt{x^2+x^{-2}+a}};$$

let $x + x^{-1} = z^{-1}$,

therefore, taking the differential coefficient of the logarithm of each side,

$$\frac{x^{-2}-1}{x+x^{-1}} = \frac{d_x z}{z},$$

$$\therefore \int_x u = \int_x \frac{d_x z}{z}\,\frac{1}{\sqrt{z^{-2}-2+a}} = \int_z \frac{1}{\sqrt{1+(a-2)z^2}},$$

a known form.

10. $$\int_x \frac{1}{(1-x^2)\sqrt{1+x^2}} = \frac{1}{\sqrt{2}}\log_\epsilon\left(\frac{\sqrt{1+x^2}+x\sqrt{2}}{\sqrt{1-x^2}}\right).$$

11. $$\int_x \frac{(x+\sqrt{1+x^2})^{\frac{m}{n}}}{\sqrt{1+x^2}}; \quad \text{let } x+\sqrt{1+x^2} = z^n;$$

$$\therefore \log_\epsilon(x+\sqrt{1+x^2}) = n\log z, \quad \therefore \frac{1}{\sqrt{1+x^2}} = \frac{nd_x z}{z},$$

$$\therefore \int_x u = \int_x \frac{nd_x z}{z}\,z^m = n\int_z z^{m-1} = \frac{n}{m}z^m = \frac{n}{m}(x+\sqrt{1+x^2})^{\frac{m}{n}}.$$

28. We shall now give instances of a few common artifices by which expressions are reduced to known forms.

I. By splitting an expression into two others of which it is the sum or difference.

1. $$\int_x \frac{a+bx}{a^2+x^2} = \int_x \left(\frac{a}{a^2+x^2} + \frac{bx}{a^2+x^2}\right)$$

$$= \int_x \frac{a}{a^2+x^2} + b\int_x \frac{x}{a^2+x^2} = \tan^{-1}\frac{x}{a} + \frac{b}{2}\log_\epsilon (a^2+x^2).$$

2. $$\int_x \frac{1 - x\cos a}{1 - 2x\cos a + x^2} = \int_x \frac{(\sin a)^2 - \cos a\,(x - \cos a)}{(\sin a)^2 + (x - \cos a)^2},$$

$$\{\text{since } (\cos a)^2 + (\sin a)^2 = 1\}$$

$$= \int_x \frac{(\sin a)^2}{(\sin a)^2 + (x - \cos a)^2} - \cos a\,.\int_x \frac{x - \cos a}{(\sin a)^2 + (x - \cos a)^2}$$

$$= \sin a\,.\tan^{-1}\frac{x - \cos a}{\sin a} - \tfrac{1}{2}\cos a\,.\log_\epsilon (1 - 2x\cos a + x^2).$$

II. By adding and subtracting the same quantity.

1. $$\int_x x\sqrt{a+x} = \int_x (x + a - a)\sqrt{a+x}$$

$$= \int_x (x+a)^{\frac{3}{2}} - a\int_x \sqrt{a+x} = \frac{2}{5}(x+a)^{\frac{5}{2}} - \frac{2a}{3}(x+a)^{\frac{3}{2}}.$$

2. $$\int_x \frac{x}{\sqrt{2ax - x^2}} = -\int_x \frac{a - x - a}{\sqrt{2ax - x^2}} = -\int_x \frac{a - x}{\sqrt{2ax - x^2}}$$

$$+ a\int_x \frac{1}{\sqrt{2ax - x^2}} = -\sqrt{2ax - x^2} + a\,\text{versin}^{-1}\frac{x}{a}.$$

3. $$\int_x \frac{x}{a + 2bx + x^2} = \int_x \frac{x + b - b}{a + 2bx + x^2}$$

$$= \int_x \frac{x + b}{a + 2bx + x^2} - b\int_x \frac{1}{a - b^2 + (x+b)^2}$$

$$= \tfrac{1}{2}\log_\epsilon (a + 2bx + x^2) - \frac{b}{\sqrt{a - b^2}}\tan^{-1}\frac{x+b}{\sqrt{a - b^2}}.$$

4. $$\int_x \frac{a+bx}{x^2-2\alpha x+\alpha^2+\beta^2} = \int_x \frac{a+b\alpha+b(x-\alpha)}{(x-\alpha)^2+\beta^2}$$

$$= \frac{a+b\alpha}{\beta}\tan^{-1}\frac{x-\alpha}{\beta} + \frac{b}{2}\log_e\{(x-\alpha)^2+\beta^2\}.$$

5. $$\int_x x^{2n-1}(a^n+x^n)^{\frac{p}{q}} = \int_x x^{n-1}(a^n+x^n-a^n)(a^n+x^n)^{\frac{p}{q}}$$

$$= \int_x x^{n-1}(a^n+x^n)^{\frac{p}{q}+1} - a^n\int_x x^{n-1}(a^n+x^n)^{\frac{p}{q}}$$

$$= \frac{q(a^n+x^n)^{\frac{p}{q}+2}}{n(p+2q)} - \frac{qa^n(a^n+x^n)^{\frac{p}{q}+1}}{n(p+q)}.$$

6. $$\int_x \frac{x^3}{(a^2+x^2)^{\frac{1}{3}}} = \frac{1}{20}(a^2+x^2)^{\frac{2}{3}}(6x^2-9a^2).$$

III. **By multiplying numerator and denominator by the same quantity.**

1. $$\int_x \frac{1}{x}\sqrt{x^2-a^2} = \int_x \frac{x^2-a^2}{x\sqrt{x^2-a^2}} = \int_x \frac{x}{\sqrt{x^2-a^2}}$$

$$- a^2\int_x \frac{1}{x\sqrt{x^2-a^2}} = \sqrt{x^2-a^2} - a\sec^{-1}\frac{x}{a}.$$

2. $$\int_x \sqrt{\frac{a+x}{a-x}} = \int_x \frac{a+x}{\sqrt{a^2-x^2}} = a\int_x \frac{1}{\sqrt{a^2-x^2}}$$

$$+ \int_x \frac{x}{\sqrt{a^2-x^2}} = a\sin^{-1}\frac{x}{a} - \sqrt{a^2-x^2}.$$

3. $$\int_x \frac{1}{\sqrt{x+a}+\sqrt{x}} = \int_x \frac{\sqrt{x+a}-\sqrt{x}}{a}$$

$$= \frac{1}{a}\int_x\sqrt{x+a} - \frac{1}{a}\int_x\sqrt{x} = \frac{2}{3a}\left\{(x+a)^{\frac{3}{2}} - x^{\frac{3}{2}}\right\}.$$

4. $$\int_x \frac{\sqrt{a^2 \pm x^2}}{x^2} = \int_x \frac{a^2 \pm x^2}{x^2\sqrt{a^2 \pm x^2}}$$

$$= \int_x a^2 x^{-3}(a^2 x^{-2} \pm 1)^{-\frac{1}{2}} \pm \int_x \frac{1}{\sqrt{a^2 \pm x^2}};$$

$$\therefore \int_x \frac{\sqrt{a^2 + x^2}}{x^2} = -\sqrt{a^2 x^{-2} + 1} + \log_\epsilon (x + \sqrt{a^2 + x^2});$$

and $$\int_x \frac{\sqrt{a^2 - x^2}}{x^2} = -\sqrt{a^2 x^{-2} - 1} - \sin^{-1}\frac{x}{a}.$$

5. $$\int_x \frac{1}{\sqrt{a^2 + x^2} - \sqrt{a^2 - x^2}} = \int_x \frac{\sqrt{a^2 + x^2} + \sqrt{a^2 - x^2}}{2x^2}$$

$$= \frac{1}{2}\int_x \frac{\sqrt{a^2 + x^2}}{x^2} + \frac{1}{2}\int_x \frac{\sqrt{a^2 - x^2}}{x^2} = -\frac{1}{2}\sqrt{a^2 x^{-2} + 1}$$

$$+ \frac{1}{2}\log_\epsilon (x + \sqrt{a^2 + x^2}) - \frac{1}{2}\sqrt{a^2 x^{-2} - 1} - \frac{1}{2}\sin^{-1}\frac{x}{a},$$ by Ex. 4.

6. $$\int_x x\sqrt{\frac{a + x}{a - x}} = \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \left(a + \frac{x}{2}\right)\sqrt{a^2 - x^2}.$$

7. $$\int_x \frac{1}{x}\sqrt{\frac{x + a}{x - a}} = \log_\epsilon (x + \sqrt{x^2 - a^2}) + \sec^{-1}\frac{x}{a}.$$

SECTION II.

RATIONAL FRACTIONS.

ART. 29. IN pursuing our object of finding $\int_x u$, we next come to the case in which u is a rational but fractional function of x; its most general form will be

$$\frac{Ax^{n-1} + Bx^{n-2} + \ldots + T}{x^n + A'x^{n-1} + \ldots + T'} = \frac{U}{V},$$

the dimension of the numerator being at least less by 1, than that of the denominator; for if not, the expression may be reduced to the sum of an integral function, and a fraction of the above form, by division. Its integration is effected by decomposing it into fractions with simpler denominators, which are called *partial fractions.*

30. Every rational fraction may be resolved into partial fractions, each of which is of one of the forms,

$$\frac{N}{x-a}, \quad \frac{M}{(x-b)^r}, \quad \frac{Kx+L}{(x-a)^2+\beta^2}, \quad \frac{Rx+S}{\{(x-a')^2+\beta'^2\}^m}.$$

Suppose the equation $x^n + A'x^{n-1} + \ldots T' = 0$ to have one real root equal to a, (r) real roots each equal to b, one pair of imaginary roots equal to $a \pm \beta\sqrt{-1}$, and (m) pairs of imaginary roots each equal to $a' \pm \beta'\sqrt{-1}$, which comprizes every case that can occur in the composition of the denominator;

$$\therefore \; V = (x-a)(x-b)^r (x^2 - 2ax + a^2 + \beta^2)(x^2 - 2a'x + a'^2 + \beta'^2)^m,$$

and $n = 1 + r + 2 + 2m$, since the dimension of each member must be the same, and V is of n dimensions.

Then $\frac{U}{V}$ may be resolved into the sum of the following fractions (all the quantities N, M_1, M_2, &c. being constants),

$$\frac{N}{x-a}+\frac{M_1}{(x-b)^r}+\frac{M_2}{(x-b)^{r-1}}+\ldots+\frac{M_r}{x-b}+\frac{Kx+L}{x^2-2\alpha x+\alpha^2+\beta^2}$$

$$+\frac{R_1x+S_1}{(x^2-2\alpha' x+\alpha'^2+\beta'^2)^m}+\frac{R_2x+S_2}{(x^2-2\alpha' x+\alpha'^2+\beta'^2)^{m-1}}\ldots\ldots$$

$$+\frac{R_mx+S_m}{x^2-2\alpha' x+\alpha'^2+\beta'^2}. \quad (1)$$

For suppose these fractions were reduced to a common denominator and added together. The common denominator would be of (n) dimensions and equal to V; and the sum of the numerators would be of $(n-1)$ dimensions, for the numerators of highest dimension would manifestly be those of the fractions whose denominators were originally of lowest dimension, and would be of $n-1$ dimensions, since these denominators are but of one. Now U is of not more than $n-1$ dimensions, and may therefore be assumed equal to the sum of the numerators; and since this equality subsists for all values of x, the coefficients of like powers in both members must be equal, and if any term in one member has no corresponding term in the other, its coefficient must be put $=0$; this will give n equations; but the number of unknown quantities is $=1+r+2+2m=n$; hence the unknown quantities are the same in number as the equations, and enter these equations only in the first degree, consequently their values will be real, and may always be assigned. If the denominator, instead of one, contain several factors of each of the above classes, then for each such factor a partial fraction, or a series of partial fractions must be introduced, similar to that belonging to the corresponding factor in the above example, and it may be shewn in the same manner that all the unknown constants can be determined. Hence it is demonstrated that, whatever be the proposed fraction, the resolution into partial fractions of the forms indicated above can be effected.

31. Hence it appears that, in the resolution of a fraction into partial fractions, for every simple factor that occurs only once in the denominator, there will be one partial fraction having that factor for its denominator, and a constant numerator; for every simple factor occurring (r) times, there will be (r) fractions having the powers of that factor from 1 to r for denominators, and constant numerators; for every irreducible quadratic factor that occurs only once, there will be a fraction having that factor for denominator, and a numerator of the form $Ax + B$; and for every irreducible quadratic factor that occurs (m) times, there will be (m) fractions having the powers of that factor from 1 to m for denominators, and numerators of the form $Rx + S$. The integration therefore of every rational fraction is reduced to that of the four forms

$$\frac{N}{x-a}, \quad \frac{M}{(x-b)^r}, \quad \frac{Kx+L}{x^2-2ax+a^2+\beta^2}, \quad \frac{Rx+S}{(x^2-2a'x+a'^2+\beta'^2)^m};$$

the three first of these have been already integrated, Arts. 12, 15, 28; and by making $x - a' = z$, the integral of the fourth becomes

$$\int_z \frac{R(z+a')+S}{(z^2+\beta'^2)^m} = \frac{R}{2}\int_z \frac{2z}{(z^2+\beta'^2)^m} + (Ra'+S)\int_z \frac{1}{(z^2+\beta'^2)^m}$$

$$= -\frac{R}{2(m-1)}\,\frac{1}{(z^2+\beta'^2)^{m-1}} + (Ra'+S)\int_z \frac{1}{(z^2+\beta'^2)^m};$$

this last integral we shall shew how to obtain in Section IV.

32. The determination of the constants N, M, &c. by the method just explained, is in general very laborious; and we shall now shew how it may be simplified for each of the four classes of factors which can compose the denominator of any fraction.

We shall shew that the fraction corresponding to any factor $x - a$ or $(x - a)^2 + \beta^2$, or the series of fractions corresponding to $(x - b)^r$ or $\{(x - a')^2 + \beta'^2\}^m$, can be separately determined; preparatory to which we may observe that if in equation (1)

Art. 30, all the fractions except the first were added together, we should have $\frac{U}{V} = \frac{N}{x-a} + \frac{P}{Q}$, P representing an integral function, and Q the quotient of V divided by $x - a$. Similarly we should have

$$\frac{U}{V} = \frac{Kx + L}{(x-a)^2 + \beta^2} + \frac{P}{Q},$$

$\frac{P}{Q}$ representing the sum of all the other partial fractions; and it is manifest that we may assume $\frac{U}{V}$ equal to the sum of any group of partial fractions which we wish to determine, and a fraction $\frac{P}{Q}$; P representing an unknown integral function, and Q the product of all the factors of V not involved in the partial fractions under consideration.

33. To determine the partial fractions corresponding to the *simple factors*, each of which occurs only *once* in the denominator.

Let these be $x - a_1$, $x - a_2, \ldots x - a_n$, therefore the denominator $V = (x - a_1)(x - a_2) \ldots (x - a_n) \cdot Q$, Q being an integral function of x which does not vanish when x equals any one of the quantities $a_1, a_2, \ldots a_n$.

$$\text{Assume } \frac{U}{V} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} \ldots + \frac{A_n}{x-a_n} + \frac{P}{Q}, \quad (1),$$

$A_1, A_2 \ldots A_n$ being constants, and P an integral function of x;

$$\therefore \frac{U}{Q} = A_1 (x - a_2) \ldots (x - a_n) + A_2 (x - a_1) \ldots (x - a_n) + \&c.$$

$$+ A_n (x - a_1) \ldots (x - a_{n-1}) + \frac{P}{Q} (x - a_1)(x - a_2) \ldots (x - a_n).$$

Now this equation subsists for all values of x, and therefore is true when $x = a_1$, a_2, &c. Let these substitutions be made successively, and let $\left(\dfrac{U}{Q}\right)_{x=a_1}$ denote the value assumed by $\dfrac{U}{Q}$, when in it, a_1 is written for x; then we have

$$\left(\frac{U}{Q}\right)_{x=a_1} = A_1 (a_1 - a_2) \dots (a_1 - a_n) \text{ which gives } A_1,$$

$$\left(\frac{U}{Q}\right)_{x=a_2} = A_2 (a_2 - a_1) \dots (a_2 - a_n) \dots\dots\dots\dots\dots A_2,$$

$$\&\text{c.} = \&\text{c.}$$

$$\left(\frac{U}{Q}\right)_{x=a_n} = A_n (a_n - a_1) \dots (a_n - a_{n-1}) \dots\dots\dots\dots\dots A_n.$$

Hence $A_1, A_2, \dots A_n$ being known, P becomes the only unknown quantity in equation (1), and may therefore be found.

Ex. 1. $$\int_x \frac{x}{x^2 + 6x + 8};$$

the roots of $x^2 + 6x + 8 = 0$, are -2, and -4,

$$\therefore x^2 + 6x + 8 = (x + 2)(x + 4);$$

$$\text{let } \frac{x}{(x + 2)(x + 4)} = \frac{A}{x + 2} + \frac{B}{x + 4},$$

$$\therefore x = A(x + 4) + B(x + 2),$$

$$\text{let } x = -2, \quad \therefore -2 = A \cdot 2, \quad \text{or } A = -1;$$

$$\text{again let } x = -4, \quad \therefore -4 = B(-2), \quad \text{or } B = 2;$$

$$\therefore \frac{x}{(x + 2)(x + 4)} = \frac{-1}{x + 2} + \frac{2}{x + 4},$$

$$\therefore \int_x \frac{x}{x^2 + 6x + 8} = -\log_\epsilon (x + 2) + 2\log_\epsilon (x + 4) = \log_\epsilon \frac{(x + 4)^2}{x + 2}.$$

Ex. 2. $\int_x \frac{x^2}{x^3+6x^2+11x+6} = \int_x \frac{x^2}{(x+1)(x+2)(x+3)}$

$$= \log_\epsilon \sqrt{\frac{(x+1)(x+3)^9}{(x+2)^8}}.$$

34. The above method requires us to know Q; in cases where Q is unknown, the following method is convenient.

Consider only one simple factor, $x-a_1$, and let

$$V = (x-a_1).S.$$

Make $\frac{U}{V} = \frac{A_1}{x-a_1} + \frac{R}{S}$, $\therefore\ U = A_1 S + R(x-a_1)$, $\therefore\ \left(\frac{U}{S}\right)_{x=a_1} = A_1$.

$$\text{But } d_x V = d_x S\,(x - a_1) + S;$$

hence, if $d_{x=a_1}V$, $S_{x=a_1}$, represent respectively the values of $d_x V$ and S when $x = a_1$, we have $S_{x=a_1} = d_{x=a_1}V$,

$$\therefore\ A_1 = \frac{U_{x=a_1}}{d_{x=a_1}V};\ \text{similarly}\ A_2 = \frac{U_{x=a_2}}{d_{x=a_2}V},\ \&\text{c.}$$

Hence the fraction corresponding to any one simple factor, is obtained without dividing V by that factor; and the decomposition of the proposed fraction may be thus represented,

$$\frac{U}{V} = \frac{U_{x=a_1}}{d_{x=a_1}V}\cdot\frac{1}{x-a_1} + \frac{U_{x=a_2}}{d_{x=a_2}V}\cdot\frac{1}{x-a_2}\ldots + \frac{U_{x=a_n}}{d_{x=a_n}V}\cdot\frac{1}{x-a_n} + \frac{P}{Q}.$$

Hence all the coefficients are known, and P becomes the only unknown quantity; Q being obtained from V by dividing it by $(x-a_1)(x-a_2)\ldots(x-a_n)$.

Ex. 1. $$\int_x \frac{x+3}{x^4-1};$$

$$\text{let } \frac{x+3}{x^4-1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{P}{x^2+1};$$

here $d_x V = 4x^3$, $\therefore A = \frac{-1+3}{-4} = -\frac{1}{2}$, $B = \frac{1+3}{4} = 1$,

$$\therefore \frac{x+3}{x^4-1} = \frac{-\frac{1}{2}}{x+1} + \frac{1}{x-1} + \frac{P}{x^2+1} = \frac{x+3}{2(x^2-1)} + \frac{P}{x^2+1};$$

$$\therefore x + 3 = \frac{x+3}{2}(x^2+1) + P(x^2-1),$$

$$\text{or } \frac{x+3}{2}(1-x^2) = P(x^2-1); \ \therefore P = -\tfrac{1}{2}(x+3);$$

$$\therefore \frac{x+3}{x^4-1} = -\frac{1}{2}\frac{1}{x+1} + \frac{1}{x-1} - \frac{1}{2}\left(\frac{x}{x^2+1} + \frac{3}{x^2+1}\right),$$

$$\therefore \int_x \frac{x+3}{x^4-1} = -\tfrac{1}{2}\log_\epsilon(x+1) + \log_\epsilon(x-1)$$
$$- \tfrac{1}{4}\log_\epsilon(x^2+1) - \tfrac{3}{2}\tan^{-1}x.$$

Ex. 2. To resolve $\frac{1}{x^3 - qx - r}$ into partial fractions, where the roots of $x^3 - qx - r = 0$ are all real but incommensurable.

If $\cos\phi = \frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}}$, the three values of x are

$$2\sqrt{\frac{q}{3}}\cos\frac{\phi}{3}, \ -2\sqrt{\frac{q}{3}}\cos\left(\frac{\pi+\phi}{3}\right), \ -2\sqrt{\frac{q}{3}}\cos\left(\frac{\pi-\phi}{3}\right);$$

let these be denoted by a, b, c, and let

$$\frac{1}{x^3 - qx - r} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}.$$

$$\therefore A = \frac{1}{d_{x=a}V} = \frac{1}{3a^2 - q} = \frac{1}{4q\left\{\left(\cos\frac{\phi}{3}\right)^2 - \frac{1}{4}\right\}}$$

$$= \frac{1}{4q \sin\frac{1}{3}(\pi + \phi) \sin\frac{1}{3}(\pi - \phi)},$$

since $\frac{1}{4} = \left(\cos\frac{\pi}{3}\right)^2$. Similarly, changing successively ϕ into $\pi + \phi$, and $\pi - \phi$, we obtain the values of B and C.

35. To determine the partial fractions corresponding to the *simple factors* which occur *more* than once in the denominator.

Let $x - a$ occur (n) times, and, therefore, $V = Q(x - a)^n$, Q being an integral function of x which does not vanish when $x = a$.

$$\text{Assume } \frac{U}{V} = \frac{A_1}{(x-a)^n} + \frac{A_2}{(x-a)^{n-1}} \dots + \frac{A_n}{x-a} + \frac{P}{Q};$$

$$\therefore\ U = A_1Q + A_2Q(x-a)\dots + A_nQ(x-a)^{n-1} + P(x-a)^n;$$

$$\text{make } x = a,\ \therefore\ \left(\frac{U}{Q}\right)_{x=a} = A_1;$$

substitute for A_1 its value, and let $\dfrac{U - A_1Q}{x - a} = U_1$, which is necessarily an integral function, because the equation

$$U - A_1Q = 0, \text{ or } A_2Q(x-a)\dots + P(x-a)^n = 0,$$

is satisfied by $x = a$;

$$\therefore\ U_1 = A_2Q\dots + A_nQ(x-a)^{n-2} + P(x-a)^{n-1};$$

$$\therefore\ \left(\frac{U_1}{Q}\right)_{x=a} = A_2: \text{ again let } \frac{U_1 - A_2Q}{x - a} = U_2;\ \therefore\ \left(\frac{U_2}{Q}\right)_{x=a} = A_3;$$

proceeding in this manner to form successively the subsidiary functions $U_3, \ldots U_{n-1}$, U_n, dividing them by Q, and then making $x = a$, we shall determine all the other quantities, viz.

$$\left(\frac{U_3}{Q}\right)_{x=a} = A_4, \text{ \&c., } \left(\frac{U_{n-1}}{Q}\right)_{x=a} = A_n, \text{ and } U_n = P.$$

Ex. 1. $$\int_x \frac{3x^2 - 1}{(x-1)^2 (x^2+1)}.$$

Let $$\frac{3x^2 - 1}{(x-1)^2 (x^2+1)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{P}{(x^2+1)};$$

$$\therefore 3x^2 - 1 = A(x^2+1) + B(x-1)(x^2+1) + P(x-1)^2,$$

$$\text{let } x = 1, \ \therefore 2 = 2A, \text{ or } A = 1;$$

$$\therefore (3x^2 - 1) - (x^2+1) = 2(x^2-1) = B(x-1)(x^2+1) + P(x-1)^2,$$

or, dividing by $x - 1$,

$$2(x+1) = B(x^2+1) + P(x-1),$$

$$\text{let } x = 1, \ \therefore 4 = 2B, \text{ or } B = 2;$$

$$\therefore 2(x+1) - 2(x^2+1) = -2x(x-1) = P(x-1), \text{ or } P = -2x,$$

$$\therefore \frac{3x^2 - 1}{(x-1)^2(x^2+1)} = \frac{1}{(x-1)^2} + \frac{2}{x-1} - \frac{2x}{x^2+1},$$

$$\therefore \int_x \frac{3x^2-1}{(x-1)^2(x^2+1)} = -\frac{1}{x-1} + \log_e \frac{(x-1)^2}{x^2+1}.$$

Ex. 2. $$\int_x \frac{x^3 + x^2 + 2}{(x+1)^2(x-1)^2 x} = -\frac{x+3}{2(x^2-1)}$$

$$+ \log_e \frac{x^2}{\sqrt[4]{(x+1)^5 (x-1)^3}}.$$

36. The above method, in order to determine any single fraction in the group, requires the computation of all the preceding ones; by the following method we can find any single fraction directly. Since

$$\frac{U}{Q} = A_1 + A_2(x-a) + A_3(x-a)^2 + \&c. + A_n(x-a)^{n-1} + X_1(x-a)^n,$$

where $X_1 = \frac{P}{Q}$, we have, by successive differentiations,

$$d_x\left(\frac{U}{Q}\right) = A_2 + 2A_3(x-a) + \&c. + (n-1)A_n(x-a)^{n-2} + X_2(x-a)^{n-1};$$

$$d_x^2\left(\frac{U}{Q}\right) = 2A_3 + \&c. + (n-1)(n-2)A_n(x-a)^{n-3} + X_3(x-a)^{n-2};$$

&c. = &c.

$$d_x^{n-1}\left(\frac{U}{Q}\right) = (n-1)(n-2)\ldots 2.1 A_n + X_n(x-a);$$

X_2, X_3,...X_n representing functions of x that do not become infinite when $x = a$. Now make $x = a$ in the above equations,

$$\therefore\ A_1 = \left(\frac{U}{Q}\right)_{x=a},\ A_2 = d_{x=a}\left(\frac{U}{Q}\right),\ A_3 = \frac{1}{1.2}\,d^2_{x=a}\left(\frac{U}{Q}\right),\ \&c. = \&c.$$

Hence the general expression for any partial fraction in the group is

$$\frac{1}{1.2.3\ldots r}\,d^r_{x=a}\left(\frac{U}{Q}\right)\cdot\frac{1}{(x-a)^{n-r}}.$$

For every factor of the same form found in the denominator V, a process similar either to this or the one in the preceding article must be gone through, to determine the partial fractions belonging to it.

37. To determine the partial fraction corresponding to a *quadratic factor* which cannot be resolved into two real simple factors, and which occurs only *once* in the denominator.

∴ X_2 X_3 & cannot ~~[illegible]~~ become ∞ when $x = a$, for $X_1 = \frac{P}{Q}$ and ∴ the denominator of $d_x X_1$ ~~can~~ only contains po[illegible] of Q & Q cannot $= (x-a)$ ∴ $d_x X_1$ never $= \infty$ and $X_2 = n X_1 + (x-a) d_x X_1$ $= n\frac{P}{Q}$ when $x = a$ and Q cannot [illegible]

Let $x^2 - ax + b$ be the quadratic factor, and $a^2 < 4b$, then the roots of the equation $x^2 - ax + b = 0$ are imaginary; also the denominator $V = (x^2 - ax + b) \,.\, Q$, where Q is an integral function.

$$\text{Assume } \frac{U}{V} = \frac{Ax + B}{x^2 - ax + b} + \frac{P}{Q},$$

A and B being constants, and P an integral function;

$$\therefore\ U - (Ax + B)\,Q = P(x^2 - ax + b) \quad (1);$$

or, separating in the first member the odd and even powers of x,

$$R(x^2) + xR'(x^2) = P(x^2 - ax + b),$$

$R(x^2)$, $R'(x^2)$, denoting integral functions of x^2, into which A and B enter only in the first degree.

In this equation make $x^2 - ax + b = 0$, that is, write $ax - b$ for x^2, then the second member vanishes, and the first may be reduced to the same form as before, $R_1(x^2) + xR_1'(x^2)$, the dimension of each function being diminished one half, and A and B still entering only in the first degree. By repeated substitutions of $ax - b$ for x^2, we shall at last arrive at a result of the form $M + Nx$, which put equal to nothing furnishes two equations $M = 0$, $N = 0$, for finding A and B; and then $P = \dfrac{U - (Ax + B)\,Q}{x^2 - ax + b}$, an integral function of x, by putting for A and B their values, and performing the division.

Ex. 1. $\displaystyle\int_x \frac{3 + 7x}{(x^2 - 2x + 5)(x^2 + x + 1)}$;

$$\text{let } \frac{3 + 7x}{(x^2 - 2x + 5)(x^2 + x + 1)} = \frac{Ax + B}{x^2 - 2x + 5} + \frac{P}{x^2 + x + 1},$$

$$\therefore\ 3 + 7x = (Ax + B)(x^2 + x + 1) + P(x^2 - 2x + 5).$$

$$\text{Make } x^2 = 2x - 5,$$

$$\therefore\ 3 + 7x = (Ax + B)(3x - 4) = A(3x^2 - 4x) + B(3x - 4);$$

or, substituting again for x^2,

$$3 + 7x = A(2x - 15) + B(3x - 4);$$

therefore, equating coefficients,

$$3 = -15A - 4B,\ 7 = 2A + 3B;\ \therefore\ A = -1,\ B = 3;$$

$$\therefore\ 3 + 7x - (3 - x)(x^2 + x + 1) = P(x^2 - 2x + 5),$$

$$\text{or } x^3 - 2x^2 + 5x = P(x^2 - 2x + 5),\ \therefore\ P = x;$$

$$\therefore\ \frac{3 + 7x}{(x^2 - 2x + 5)(x^2 + x + 1)} = \frac{3 - x}{x^2 - 2x + 5} + \frac{x}{x^2 + x + 1}$$

$$= \frac{2}{(x-1)^2 + 4} - \frac{x - 1}{x^2 - 2x + 5} + \frac{x + \frac{1}{2}}{x^2 + x + 1} - \frac{1}{2}\frac{1}{(x + \frac{1}{2})^2 + \frac{3}{4}},$$

$$\therefore\ \int_x \frac{3 + 7x}{(x^2 - 2x + 5)(x^2 + x + 1)} = \tan^{-1}\frac{x - 1}{2} - \tfrac{1}{2}\log_\epsilon(x^2 - 2x + 5)$$

$$+ \tfrac{1}{2}\log_\epsilon(x^2 + x + 1) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x + 1}{\sqrt{3}}.$$

Ex. 2. $$\int_x \frac{1}{(x^3 + 1)(x^2 + 1)};$$

$$\text{let } \frac{1}{(x^3 + 1)(x^2 + 1)} = \frac{Ax + B}{x^2 - x + 1} + \frac{A_1x + B_1}{x^2 + 1} + \frac{C}{x + 1},$$

$$\therefore\ 1 = (Ax + B)(x^2 + 1)(x + 1) + (A_1x + B_1)(x^3 + 1)$$
$$+ C(x^2 + 1)(x^2 - x + 1).$$

$$\text{Make } x = -1,\ \therefore\ 1 = 6C;$$

make $x^2 + 1 = 0$, or $x^2 = -1$, $\therefore\ 1 = (A_1x + B_1)(-x + 1)$

$$= A_1(-x^2 + x) + B_1(-x + 1) = A_1(1 + x) + B_1(-x + 1),$$

$$\therefore\ A_1 = B_1 = \tfrac{1}{2}.$$

Make $x^2 - x + 1 = 0$,

$$\therefore\ 1=(Ax+B)x(x+1)=(Ax+B)(2x-1)=A(2x^2-x)+B(2x-1),$$

$$\text{or } 1 = A(x-2) + B(2x-1);$$

$$\therefore\ 1 = -2A - B,\quad 0 = A + 2B;\quad A = -\tfrac{2}{3},\quad B = \tfrac{1}{3};$$

$$\therefore\ \frac{1}{(x^3+1)(x^2+1)} = \tfrac{1}{3}\cdot\frac{1-2x}{x^2-x+1} + \tfrac{1}{2}\cdot\frac{1+x}{x^2+1} + \tfrac{1}{6}\cdot\frac{1}{x+1};$$

$$\therefore\ \int_x \frac{1}{(x^3+1)(x^2+1)} = \log_\epsilon\left(\frac{(x^2+1)^{\frac{1}{4}}(x+1)^{\frac{1}{6}}}{(x^2-x+1)^{\frac{1}{3}}}\right) + \tfrac{1}{2}\tan^{-1}x.$$

38. In the above method, the division of V by x^2-ax+b, in order to find Q, will sometimes be a tedious process and may be avoided in the following manner.

Since $V=(x^2-ax+b)\,Q$,

$$\therefore\ d_xV = d_xQ\,.\,(x^2-ax+b) + Q\,.\,(2x-a);$$

hence when $ax-b$ is substituted for x^2, d_xV and $Q\,.\,(2x-a)$ have the same value; and therefore when Q is unknown, A and B may be more conveniently obtained from reducing, by repeated substitutions,

$$(2x-a)\,.\,U - (Ax+B)\,d_xV \text{ to the form } M + Nx,$$

than, in the usual way, by reducing $U - (Ax+B)\,Q$.

Ex. In the resolution of $\dfrac{1}{x^8+x^7-x^4-x^3}$ into partial fractions, to find that whose denominator is x^2+1.

The quantity to be reduced by successive substitutions of -1 for x^2, is

$$2x - (Ax+B)(8x^7+7x^6-4x^3-3x^2), \text{ which becomes}$$

$$2x-(Ax+B)\{-(8x+7)+4x+3\} = 2x+4(Ax+B)+4(-A+Bx);$$

$$\therefore\ 2\overset{\checkmark}{+}4B+4A=0,\quad -4A+4B=0;\quad \therefore\ A=B=-\tfrac{1}{4};$$

and the required fraction is $-\frac{1}{4}\frac{x+1}{x^2+1}$.

39. To determine the partial fractions corresponding to a quadratic factor, not capable of being resolved into two real simple factors, which occurs in the denominator (n) times.

Here $V=(x^2-ax+b)^n Q$, where Q is an integral function of x.

Assume $$\frac{U}{V}=\frac{A_1x+B_1}{(x^2-ax+b)^n}+\frac{A_2x+B_2}{(x^2-ax+b)^{n-1}}$$
$$\ldots+\frac{A_nx+B_n}{x^2-ax+b}+\frac{P}{Q},$$

A_1, B_1,...A_n, B_n, being all constants, and P an integral function of x;

$$\therefore\ U-(A_1x+B_1)\,Q=(A_2x+B_2)\,(x^2-ax+b)\,Q$$
$$\ldots+(A_nx+B_n)\,(x^2-ax+b)^{n-1}\,Q+P\,(x^2-ax+b)^n.$$

In this equation, substitute $ax-b$ for x^2; then the second member vanishes; and the first, by repeated substitutions of $ax-b$ for x^2, can be reduced to the form M_1+N_1x, (as shewn in article 37.) which put $=0$, gives $M_1=0$, $N_1=0$; and since A_1, B_1 enter these equations only in the first degree, they can always be determined. Substituting these values of A_1 and B_1, and performing the division, we have $\frac{U-(A_1x+B_1)Q}{x^2-ax+b}=U_1$, an integral function.

$$\therefore\ U_1=(A_2x+B_2)\overset{Q}{\wedge}+\text{&c.}+(A_nx+B_n)\,(x^2-ax+b)^{n-2}\,Q$$
$$+P\,(x^2-ax+b)^{n-1};$$

and substituting for x^2, and proceeding as above, we find $M_2+N_2x=0$, which gives A_2 and B_2.

Similarly, we can determine A_3, B_3, &c.; and $U_n = P$.

Ex. To resolve $\dfrac{2x^5 - x^2}{(x^2+1)^2(x^2+x+1)^2}$, into partial fractions.

$$\text{Let } \frac{2x^5 - x^2}{(x^2+1)^2(x^2+x+1)^2} = \frac{Ax+B}{(x^2+1)^2} + \frac{A_1x+B_1}{x^2+1} + \frac{P}{(x^2+x+1)^2},$$

$$\therefore\ x^2(2x^3-1) = (Ax+B)(x^2+x+1)^2$$

$$+ (A_1x+B_1)(x^2+1)(x^2+x+1)^2 + P(x^2+1)^2; \quad (1),$$

$$\text{let } x^2+1=0, \quad \therefore\ 2x+1 = -(Ax+B),$$

$$\therefore\ A=-2, \quad B=-1;$$

hence, substituting these values in equation (1),

$$x^2(2x^3-1) + (1+2x)(x^2+x+1)^2$$

$$= (A_1x+B_1)(x^2+1)(x^2+x+1)^2 + P(x^2+1)^2;$$

but the first member of this equation, since it must have x^2+1 for a factor, may be put under the form

$$x^2(2x^3-1+2x+1) + (1+2x)\left\{(x^2+1)^2 + 2x(x^2+1)\right\}$$

$$= 2x^3(x^2+1) + (2x+1)(x+1)^2(x^2+1),$$

$$\therefore\ 2x^3 + (2x+1)(x+1)^2 = (A_1x+B_1)(x^2+x+1)^2 + P(x^2+1);$$

$$\text{let } x^2+1=0, \quad \therefore\ -2x + (2x+1)\,2x = (A_1x+B_1)\,x^2,$$

$$\text{or } -4 = -(A_1x+B_1),$$

$$\therefore\ A_1=0, \quad B_1=4;$$

$$\therefore\ 2x^3 + (2x+1)(x^2+1+2x) - 4(x^2+x+1)^2 = P(x^2+1),$$

$$\text{or } 2x(x^2+1) + (2x+1)(x^2+1) - 4(x^2+1)^2 - 8x(x^2+1)$$

$$= P(x^2+1),$$

$$\therefore\ P = -4x^2 - 4x - 3\,;$$

$$\therefore\ \frac{2x^5 - x^2}{(x^2+1)^2\,(x^2+x+1)^2} = -\frac{1+2x}{(x^2+1)^2} + \frac{4}{x^2+1} - \frac{4x^2+4x+3}{(x^2+x+1)^2}$$

$$= -\frac{1+2x}{(x^2+1)^2} + \frac{4}{x^2+1} - \frac{4}{(x^2+x+1)} + \frac{1}{(x^2+x+1)^2}\,.$$

We shall now proceed to the consideration of certain other fractions which merit attention, both because they are the forms to which many expressions can be reduced, and because they can be integrated by elegant particular methods more easily than by the general ones just explained.

40. To integrate $\dfrac{1}{1-x^n}$.

Case I. Let (n) be even; then resolving $1 - x^n$ into its quadratic factors, we have

$$1 - x^n = (1 - x^2)\left(1 - 2x\cos\frac{2\pi}{n} + x^2\right)$$

$$\times\left(1 - 2x\cos\frac{4\pi}{n} + x^2\right)\ldots\ldots\ldots\ldots\left(1 - 2x\cos\frac{(n-2)\pi}{n} + x^2\right);$$

therefore, taking the Naperian logarithms of both sides, and differentiating,

$$\frac{-nx^{n-1}}{1-x^n} = \frac{-2x}{1-x^2} + \frac{-2\cos\dfrac{2\pi}{n} + 2x}{1 - 2x\cos\dfrac{2\pi}{n} + x^2}$$

$$+\frac{-2\cos\dfrac{4\pi}{n} + 2x}{1 - 2x\cos\dfrac{4\pi}{n} + x^2}\ldots\ldots + \frac{-2\cos\dfrac{(n-2)\pi}{n} + 2x}{1 - 2x\cos\dfrac{(n-2)\pi}{n} + x^2}.$$

Multiply by x, and subtract each side from n, that is, the first from n, and each term of the second from 2, since the number of terms is $\frac{n}{2}$, and divide by n; then

$$\frac{1}{1-x^n} = \frac{2}{n}\left\{\frac{1}{1-x^2} + \frac{1 - x\cos\frac{2\pi}{n}}{1 - 2x\cos\frac{2\pi}{n} + x^2}\right.$$

$$\left. + \frac{1 - x\cos\frac{4\pi}{n}}{1 - 2x\cos\frac{4\pi}{n} + x^2} \ldots\ldots + \frac{1 - x\cos\frac{(n-2)\pi}{n}}{1 - 2x\cos\frac{(n-2)\pi}{n} + x^2}\right\}.$$

Case II. Let n be odd, then the resolution of $1 - x^n$ into factors, gives

$$1 - x^n = (1 - x)\left(1 - 2x\cos\frac{2\pi}{n} + x^2\right)$$

$$\times\left(1 - 2x\cos\frac{4\pi}{n} + x^2\right)\ldots\ldots\ldots\left(1 - 2x\cos\frac{(n-1)\pi}{n} + x^2\right).$$

Therefore, taking the Naperian logarithms of both sides, and differentiating,

$$\frac{-nx^{n-1}}{1-x^n} = \frac{-1}{1-x} + \frac{-2\cos\frac{2\pi}{n} + 2x}{1 - 2x\cos\frac{2\pi}{n} + x^2} + \&c.$$

$$+ \frac{-2\cos\frac{(n-1)\pi}{n} + 2x}{1 - 2x\cos\frac{(n-1)\pi}{n} + x^2}.$$

Multiply by x, and subtract each side from n, that is, the first from n, and the first term of the other side from 1, and each of the remaining $\frac{n-1}{2}$ terms from 2, then

$$\frac{1}{1-x^n} = \frac{2}{n}\left\{\frac{\frac{1}{2}}{1-x} + \frac{1 - x\cos\frac{2\pi}{n}}{1 - 2x\cos\frac{2\pi}{n} + x^2}\right.$$

$$\left. + \frac{1 - x\cos\frac{4\pi}{n}}{1 - 2x\cos\frac{4\pi}{n} + x^2} \ldots\ldots + \frac{1 - x\cos\frac{(n-1)\pi}{n}}{1 - 2x\cos\frac{(n-1)\pi}{n} + x^2}\right\}.$$

Hence $\int_x \frac{1}{1-x^n}$ can be found; for each of its partial fractions is of the form $\frac{1 - x\cos\alpha}{1 - 2x\cos\alpha + x^2}$, the integral of which is (Ex. 2. Art. 28.)

$$\sin\alpha \,.\, \tan^{-1}\frac{x - \cos\alpha}{\sin\alpha} - \tfrac{1}{2}\cos\alpha \,.\, \log_e(1 - 2x\cos\alpha + x^2).$$

Ex. $$\int_x \frac{1}{1-x^6};$$

$$1 - x^6 = (1 - x^2)\left(1 - 2x\cos\frac{\pi}{3} + x^2\right)\left(1 - 2x\cos\frac{2\pi}{3} + x^2\right);$$

therefore, taking logarithms and differentiating,

$$\frac{-6x^5}{1-x^6} = \frac{-2x}{1-x^2} + \frac{-2\cos\frac{\pi}{3} + 2x}{1 - 2x\cos\frac{\pi}{3} + x^2} + \frac{-2\cos\frac{2\pi}{3} + 2x}{1 - 2x\cos\frac{2\pi}{3} + x^2}.$$

Multiply both sides by x, subtract the first side from 6, and each term of the second from 2, and divide by 6;

$$\therefore\ \frac{1}{1-x^6} = \frac{1}{3}\left\{\frac{1}{1-x^2} + \frac{1 - x\cos\frac{\pi}{3}}{1 - 2x\cos\frac{\pi}{3} + x^2} + \frac{1 - x\cos\frac{2\pi}{3}}{1 - 2x\cos\frac{2\pi}{3} + x^2}\right\}.$$

$$\text{Now } \int_x \frac{1}{1-x^2} = \tfrac{1}{2}\log_\epsilon \frac{1+x}{1-x};$$

$$\int_x \frac{1 - x\cos\frac{\pi}{3}}{1 - 2x\cos\frac{\pi}{3} + x^2} = \frac{\sqrt{3}}{2}\tan^{-1}\frac{2x-1}{\sqrt{3}} - \tfrac{1}{4}\log_\epsilon(1 - x + x^2),$$

$$\int_x \frac{1 - x\cos\frac{2\pi}{3}}{1 - 2x\cos\frac{2\pi}{3} + x^2} = \frac{\sqrt{3}}{2}\tan^{-1}\frac{2x+1}{\sqrt{3}} + \tfrac{1}{4}\log_\epsilon(1+x+x^2),$$

$$\because\ \cos\frac{\pi}{3} = \tfrac{1}{2},\ \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},\ \cos\frac{2\pi}{3} = -\tfrac{1}{2},\ \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2};$$

therefore, collecting the results,

$$\int_x \frac{1}{1-x^6} = \frac{1}{3}\left\{\tfrac{1}{2}\log_\epsilon\left(\frac{1+x}{1-x}\right) + \tfrac{1}{4}\log_\epsilon\left(\frac{1+x+x^2}{1-x+x^2}\right)\right.$$

$$\left. + \frac{\sqrt{3}}{2}\left(\tan^{-1}\frac{2x+1}{\sqrt{3}} + \tan^{-1}\frac{2x-1}{\sqrt{3}}\right)\right\}$$

$$= \frac{1}{6}\log_\epsilon\left(\frac{1+x}{1-x}\cdot\sqrt{\frac{1+x+x^2}{1-x+x^2}}\right) + \frac{1}{2\sqrt{3}}\tan^{-1}\frac{x\sqrt{3}}{1-x^2}.$$

41. To integrate $\dfrac{1}{1+x^n}$.

Case I. Let n be even;

$$\therefore\ 1 + x^n = \left(1 - 2x\cos\frac{\pi}{n} + x^2\right)$$

$$\times\ \left(1 - 2x\cos\frac{3\pi}{n} + x^2\right)\ldots\ldots\left(1 - 2x\cdot\cos\frac{(n-1)\pi}{n} + x^2\right);$$

hence, by the same process as for $\dfrac{1}{1-x^n}$ when n is even, we find

$$\frac{1}{1+x^n} = \frac{2}{n}\left\{\frac{1 - x\cos\frac{\pi}{n}}{1 - 2x\cos\frac{\pi}{n} + x^2} + \frac{1 - x\cos\frac{3\pi}{n}}{1 - 2x\cos\frac{3\pi}{n} + x^2}\right.$$

$$\left.\ldots\ldots + \frac{1 - x\cos\frac{(n-1)\pi}{n}}{1 - 2x\cos\frac{(n-1)\pi}{n} + x^2}\right\}.$$

Case II. Let n be odd;

$$\therefore\ 1 + x^n = (1 + x)\left(1 - 2x\cos\frac{\pi}{n} + x^2\right)$$

$$\times\ \left(1 - 2x\cos\frac{3\pi}{n} + x^2\right)\ldots\ldots\left(1 - 2x\cos\frac{(n-2)\pi}{n} + x^2\right);$$

and by the same process as for $\dfrac{1}{1-x^n}$ when n is odd, we find

$$\frac{1}{1+x^n} = \frac{2}{n}\left\{\frac{\frac{1}{2}}{1+x} + \frac{1 - x\cos\frac{\pi}{n}}{1 - 2x\cos\frac{\pi}{n} + x^2}\right.$$

$$\left.+ \frac{1 - x\cos\frac{3\pi}{n}}{1 - 2x\cos\frac{3\pi}{n} + x^2}\ldots\ldots + \frac{1 - x\cos\frac{(n-2)\pi}{n}}{1 - 2x\cos\frac{(n-2)\pi}{n} + x^2}\right\}.$$

Hence $\int_x \frac{1}{1+x^n}$ can be obtained.

Ex.
$$\int_x \frac{1}{1+x^4} = \frac{1}{4\sqrt{2}} \log_\epsilon \frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}$$
$$+\frac{1}{2\sqrt{2}} \tan^{-1} \frac{x\sqrt{2}}{1-x^2}.$$

42. To integrate $\frac{x^m}{x^n \pm 1}$.

Let $x^2 - 2x\cos\phi + 1 = (x-a)(x-b)$ represent any quadratic factor of the denominator V, $\frac{A}{x-a} + \frac{B}{x-b}$ the fractions corresponding to it;

$\therefore$ (Art. 34.) $A = \frac{U_{x=a}}{d_{x=a}V} = \frac{a^m}{na^{n-1}} = \frac{a^{m+1}}{\mp n}$; similarly, $B = \frac{b^{m+1}}{\mp n}$,

since a and b are roots of $x^n \pm 1 = 0$;

$$\therefore \frac{A}{x-a} + \frac{B}{x-b} = \frac{1}{\mp n}\left(\frac{a^{m+1}}{x-a} + \frac{b^{m+1}}{x-b}\right)$$

$$= \frac{1}{\mp n}\left(\frac{x(a^{m+1}+b^{m+1}) - ab(a^m+b^m)}{x^2 - 2x\cos\phi + 1}\right),$$

$$= \frac{1}{\mp n}\left(\frac{2x\cos(m+1)\phi - 2\cos m\phi}{x^2 - 2x\cos\phi + 1}\right),$$

since $a + b = 2\cos\phi$, $ab = 1$.

This is the general form of the partial fraction, which may also be expressed by

$$\frac{1}{\mp n}\left(\frac{2x\cos(m+1)\phi - 2\cos(m+1)\phi\cos\phi - 2\sin(m+1)\phi\sin\phi}{x^2 - 2x\cos\phi + 1}\right),$$

$$\text{or } \frac{1}{\mp n}\left(\cos(m+1)\phi\,\frac{2x-2\cos\phi}{x^2-2x\cos\phi+1}-2\sin(m+1)\phi\right.$$

$$\left.\cdot\,\frac{\sin\phi}{(x-\cos\phi)^2+(\sin\phi)^2}\right).$$

Hence the general term of $\int_x \frac{x^m}{x^n \pm 1}$ is

$$\frac{1}{\mp n}\left\{\cos(m+1)\phi.\log_\epsilon(x^2-2x\cos\phi+1)\right.$$

$$\left.-2\sin(m+1)\phi.\tan^{-1}\frac{x-\cos\phi}{\sin\phi}\right\}.$$

Case I. Take the upper sign, then $\phi = \frac{(2\lambda+1)\pi}{n}$, λ representing an integer; and to obtain all the terms, λ must be taken from 0 to $\frac{n}{2}-1$, when n is even, and to $\frac{n-3}{2}$ when n is odd; the remaining simple factor, $x+1$, in the latter case giving the partial fraction $\frac{(-1)^m}{n}\cdot\frac{1}{x+1}$, and consequently, the integral $\frac{(-1)^m}{n}\log_\epsilon(x+1)$.

Case II. Take the lower sign, then $\phi = \frac{2\lambda\pi}{n}$; and when n is even, λ must be taken from 1 to $\frac{n}{2}-1$, the remaining quadratic factor being x^2-1, which gives the partial fractions

$$\frac{1}{n}\left(\frac{1}{x-1}+\frac{(-1)^{m+1}}{x+1}\right),$$

and, therefore, the integrals $\frac{1}{n}\log_\epsilon(x-1)+\frac{(-1)^{m+1}}{n}\log_\epsilon(x+1)$.

When n is odd, λ must be taken from 1 to $\frac{n-1}{2}$, the remaining factor, $x-1$, giving the partial fraction $\frac{1}{n}\cdot\frac{1}{x-1}$, and, therefore, the integral $\frac{1}{n}\log_\epsilon(x-1)$.

43. Hence we can integrate $\frac{x^m}{a+bx^n}$;

for, making $\frac{bx^n}{a} = z^n$ and therefore $x^m = \left(\frac{a}{b}\right)^{\frac{m+1}{n}} z^m d_x z$, we have

$$\int_x \frac{x^m}{a+bx^n} = \frac{1}{a}\left(\frac{a}{b}\right)^{\frac{m+1}{n}} \int_x \frac{z^m d_x z}{1+z^n} = \frac{1}{a}\left(\frac{a}{b}\right)^{\frac{m+1}{n}} \int_z \frac{z^m}{1+z^n}.$$

When n is even, and a and b have different signs, we must assume $-\frac{bx^n}{a} = z^n$.

44. To integrate $\frac{x^{m-1} \pm x^{n-m-1}}{x^n+1}$.

Let $x^2 - 2x\cos\phi + 1 = (x-a)(x-b)$ be any quadratic factor of the denominator V, $\frac{A}{x-a} + \frac{B}{x-b}$ the fractions corresponding to it.

Case I. Take the upper sign;

$$\therefore A = \frac{U_{x=a}}{d_{x=a}V} = \frac{a^{m-1}+a^{n-m-1}}{na^{n-1}} = \frac{a^m - a^{-m}}{-n} = \frac{a^m - b^m}{-n};$$

$$\text{similarly } B = \frac{b^m - a^m}{-n};$$

$$\therefore \frac{A}{x-a} + \frac{B}{x-b} = \frac{a^m - b^m}{n}\left\{-\frac{1}{x-a} + \frac{1}{x-b}\right\}$$

$$= \frac{-1}{n}\cdot\frac{(a^m-b^m)(a-b)}{x^2-2x\cos\phi+1} = \frac{4}{n}\cdot\frac{a^m-b^m}{2\sqrt{-1}}\cdot\frac{a-b}{2\sqrt{-1}}\cdot\frac{1}{x^2-2x\cos\phi+1}$$

$$= \frac{4}{n}\sin m\phi \sin\phi \frac{1}{x^2-2x\cos\phi+1},$$

which is the general form of the partial fraction.

Hence, the general term of $\int_x \frac{x^{m-1} + x^{n-m-1}}{x^n + 1}$ is

$$\frac{4}{n} \sin m\phi \,.\, \tan^{-1} \frac{x - \cos\phi}{\sin\phi}, \text{ where } \phi = \frac{(2\lambda + 1)\pi}{n};$$

and to obtain all the terms, λ must be taken from 0 to $\frac{n}{2} - 1$, when n is even, and to $\frac{n-3}{2}$ when n is odd; the remaining simple factor $x + 1$ in the latter case giving the partial fraction $\frac{1}{x+1}\left(\frac{(-1)^m + (-1)^{n-m}}{-n}\right)$, which vanishes;

$$\therefore \int_x \frac{x^{m-1} + x^{n-m-1}}{x^n + 1} = \frac{4}{n}\left\{\sin\frac{m\pi}{n} \,.\, \tan^{-1}\frac{x - \cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}\right.$$

$$\left. + \sin\frac{3m\pi}{n} \,.\, \tan^{-1}\frac{x - \cos\frac{3\pi}{n}}{\sin\frac{3\pi}{n}} + \&c.\right\},$$

the number of terms being $\frac{n}{2}$ or $\frac{n-1}{2}$, according as n is even or odd.

Case II. Take the lower sign, then

$$A = \frac{a^{m-1} - a^{n-m-1}}{na^{n-1}} = \frac{a^m + a^{-m}}{-n} = \frac{a^m + b^m}{-n};$$

this must also be the value of B since a, b, are similarly involved;

$$\therefore \frac{A}{x - a} + \frac{B}{x - b} = \frac{a^m + b^m}{-n}\left(\frac{1}{x-a} + \frac{1}{x-b}\right)$$

$$= \frac{2\cos m\phi}{-n}\left(\frac{2x - 2\cos\phi}{x^2 - 2x\cos\phi + 1}\right),$$

the general form of the partial fraction, the integral of which is

$$-\frac{2}{n}\cos m\phi \,.\, \log_\epsilon (x^2 - 2x\cos\phi + 1);$$

$$\therefore \int_x \frac{x^{m-1} - x^{n-m-1}}{x^n + 1} = -\frac{2}{n}\left\{\cos\frac{m\pi}{n} \,.\, \log_\epsilon\left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\right.$$

$$\left. + \cos\frac{3m\pi}{n} \,.\, \log_\epsilon\left(x^2 - 2x\cos\frac{3\pi}{n} + 1\right) + \&c.\right\}.$$

The series within the brackets goes on to $\frac{n}{2}$ terms, if n be even, and to $\frac{n-1}{2}$ terms, if odd; the simple factor, in the latter case, giving the additional term $(-1)^{m+1}\frac{2}{n}\log_\epsilon (x+1)$.

45. In the same manner may $\int_x \frac{x^{m-1} \pm x^{n-m-1}}{x^n - 1}$ be obtained, in logarithms only, or circular arcs only, according as the upper or lower sign is taken.

46. To integrate $\frac{x^m}{x^{2n} - 2\cos\theta\, x^n + 1}$.

Let $x^2 - 2x\cos\phi + 1 = (x-a)(x-b)$ be any quadratic factor of the denominator V,

$$\therefore\ a^n = \cos\theta + \sqrt{-1}\,\sin\theta, \qquad b^n = \cos\theta - \sqrt{-1}\,\sin\theta;$$

and let $\frac{A}{x-a} + \frac{B}{x-b}$ be the corresponding fractions,

$$\therefore\ A = \frac{U_{x=a}}{d_{x=a}V} = \frac{a^m}{2n\,(a^n - \cos\theta)\,a^{n-1}} = \frac{1}{2n\sqrt{-1}\,\sin\theta\, a^{n-m-1}}$$

$= \frac{b^{n-m-1}}{2n\sqrt{-1}\,\sin\theta}$, since $ab = 1$; similarly $B = -\frac{a^{n-m-1}}{2n\sqrt{-1}\,\sin\theta}$;

$$\therefore \frac{A}{x-a} + \frac{B}{x-b} = \frac{1}{2n\sqrt{-1}\sin\theta}\left(\frac{b^{n-m-1}}{x-a} - \frac{a^{n-m-1}}{x-b}\right)$$

$$= \frac{1}{2n\sqrt{-1}\sin\theta}\,\frac{a^{n-m} - b^{n-m} - x\,(a^{n-m-1} - b^{n-m-1})}{x^2 - 2x\cos\phi + 1}$$

$$= \frac{1}{n\sin\theta}\,\frac{\sin(n-m)\,\phi - x\sin(n-m-1)\,\phi}{x^2 - 2x\cos\phi + 1},$$

which is the general form of the partial fraction.

Hence the general term of $\int_x \dfrac{x^m}{x^{2n} - 2\cos\theta x^n + 1}$ is

$$\frac{1}{n\sin\theta}\left\{\cos(n-m-1)\,\phi\,.\tan^{-1}\frac{x-\cos\phi}{\sin\phi}\right.$$
$$\left. - \tfrac{1}{2}\sin(n-m-1)\,\phi\,.\log_\epsilon(x^2 - 2x\cos\phi + 1) + C\right\},$$

where $\phi = \dfrac{2\lambda\pi + \theta}{n}$; and to obtain all the terms, λ must be taken from 0 to $n-1$.

47. If in the preceding article we suppose

$$C = \cos(n-m-1)\,\phi\,.\tan^{-1}\frac{\cos\phi}{\sin\phi},$$

$$\text{then } \tan^{-1}\frac{x-\cos\phi}{\sin\phi} + \tan^{-1}\frac{\cos\phi}{\sin\phi}$$

$$= \tan^{-1}\frac{\dfrac{x}{\sin\phi}}{1 - \dfrac{x-\cos\phi}{\sin\phi}\,.\,\dfrac{\cos\phi}{\sin\phi}} = \tan^{-1}\frac{x\sin\phi}{1 - x\cos\phi};$$

and therefore the general term of the integral becomes

$$\frac{1}{n\sin\theta}\left\{\cos(n-m-1)\,\phi\,.\tan^{-1}\frac{x\sin\phi}{1-x\cos\phi}\right.$$
$$\left. - \tfrac{1}{2}\sin(n-m-1)\,\phi\,.\log_\epsilon(x^2 - 2x\cos\phi + 1)\right\}.$$

By this determination of the constant, the integral vanishes when $x = 0$; here also we are furnished with an instance of the change made in the form of the integral, by the introduction of a constant of a particular form.

48. Hence we can integrate $\dfrac{x^m}{a + bx^n + cx^{2n}}$.

Let $\dfrac{c}{a} x^{2n} = z^{2n}$, $\therefore\ x^n = \left(\dfrac{a}{c}\right)^{\frac{1}{2}} z^n$, and $x^m = \left(\dfrac{a}{c}\right)^{\frac{m+1}{2n}} z^m d_z z$,

$$\therefore\ \int_x \frac{x^m}{a + bx^n + cx^{2n}} = \left(\frac{a}{c}\right)^{\frac{m+1}{2n}} \int_z \frac{z^m d_z z}{a\left(1 + \dfrac{b}{\sqrt{ac}} z^n + z^{2n}\right)}$$

$$= \frac{1}{a} \cdot \left(\frac{a}{c}\right)^{\frac{m+1}{2n}} \int_z \frac{z^m}{1 - 2\cos\theta z^n + z^{2n}},$$

if $\cos\theta = -\dfrac{b}{2\sqrt{ac}}$, which requires that $4ac$ be greater than b^2.

But if $4ac$ be not greater than b^2, then the values of x^n in the equation $cx^{2n} + bx^n + a = 0$ are real; let them be denoted by f and g,

$$\therefore\ \frac{1}{a + bx^n + cx^{2n}} = \frac{1}{c(x^n - f)(x^n - g)}$$

$$= \frac{1}{c(f-g)} \left\{ \frac{1}{x^n - f} - \frac{1}{x^n - g} \right\};$$

$$\therefore\ \int_x \frac{x^m}{a + bx^n + cx^{2n}} = \frac{1}{c(f-g)} \left\{ \int_x \frac{x^m}{x^n - f} - \int_x \frac{x^m}{x^n - g} \right\},$$

and is reduced to Art. 42.

SECTION III.

RATIONALIZATION.

R

ART. 49. HAVING proved that $\int_x u$ can always be found when u is a rational function of x, any proposed function must be considered as integrated, when by any transformation it can be reduced to the form $\int_z R(z)$, where $R(z)$ denotes a rational function of z, or to such an irrational form as admits of being rationalized. This transformation can be effected only in particular cases depending upon the form of u; the following are among the most useful.

I. To rationalize $\int_x R\left\{x^r, \left(\dfrac{a+bx}{a_1+b_1x}\right)^{\frac{m}{n}}, \left(\dfrac{a+bx}{a_1+b_1x}\right)^{\frac{p}{q}}, \text{\&c.}\right\}$

Assume $\dfrac{a+bx}{a_1+b_1x} = z^l$, where $l = nq \ldots =$ product of denominators of the indices,

$$\therefore\ a+bx = a_1z^l + b_1xz^l \quad \text{or} \quad x = \frac{a_1z^l - a}{b - b_1z^l},$$

$$\therefore\ d_zx = \frac{lz^{l-1}(a_1b - ab_1)}{(b - b_1z^l)^2},$$

also $\left(\dfrac{a+bx}{a_1+b_1x}\right)^{\frac{m}{n}} = z^{\frac{lm}{n}}$, $\left(\dfrac{a+bx}{a_1+b_1x}\right)^{\frac{p}{q}} = z^{\frac{lp}{q}}$, &c. which are all rational.

Hence by substituting these values, $\int_x u = \int_z u\,d_zx$ becomes of the form $\int_z R(z)$.

It is manifest that $\int_x R\,(x,\ x^{\frac{m}{n}},\ x^{\frac{p}{q}},\ \&c.)$

$$\text{and } \int_x R\,\{x,\ (a+bx)^{\frac{m}{n}},\ (a+bx)^{\frac{p}{q}},\ \&c.\}$$

are particular cases of the above; and to rationalize them the assumptions are $x = z^{nq\cdots}$, and $a + bx = z^{nq\cdots}$, respectively.

$$\text{Also } \int_x x^{n-1} R\left\{x^{mn},\ \left(\frac{a+bx^n}{a_1+b_1x^n}\right)^{\frac{p}{q}},\ \left(\frac{a+bx^n}{a_1+b_1x^n}\right)^{\frac{r}{s}},\ \&c.\right\}$$

is immediately reduced to the above form by making $x^n = z$.

Ex. 1. $\displaystyle\int_x \frac{1 + x^{\frac{1}{2}} + x^{\frac{2}{3}}}{1 + x^{\frac{1}{3}}}$; let $x = z^6$, $\therefore\ d_z x = 6z^5$,

$$\therefore\ \int_x u = \int_z \frac{1 + z^3 + z^4}{1 + z^2}\,6z^5 = \text{(by division)}$$

$$6\int_z \left(z^7 + z^6 - z^5 - z^4 + 2z^3 + z^2 - 2z - 1 + \frac{2z+1}{1+z^2}\right)$$

$$= 6\left(\frac{z^8}{8} + \frac{z^7}{7} - \frac{z^6}{6} - \frac{z^5}{5} + \frac{z^4}{2} + \frac{z^3}{3} - z^2 - z + \log_e(1+z^2) + \tan^{-1} z\right).$$

Ex. 2. $\displaystyle\int_x \frac{x}{(1+x)^{\frac{1}{3}} - (1+x)^{\frac{1}{2}}}$; let $1 + x = z^6$,

$$\therefore\ \int_x u = \int_z \frac{-6z^3(1-z^6)}{1-z} = -\left(\tfrac{3}{2}z^4 + \tfrac{6}{5}z^5 + z^6 + \tfrac{6}{7}z^7 + \tfrac{3}{4}z^8 + \tfrac{2}{3}z^9\right),$$

by dividing and integrating.

Ex. 3. $\displaystyle\int_x \frac{1}{(c + ex^2)\sqrt{a+bx}}$;

$$\text{let } \sqrt{a+bx} = z,\quad \therefore\ \frac{1}{\sqrt{a+bx}} = \frac{2}{b}\,d_x z,$$

SECTION IV.

FORMULÆ OF REDUCTION.

51. We next come to the case either where we are unable to rationalize $\int_x u$, or where that would be a disadvantageous practical method of effecting the integration; but where, by means of certain assumptions, a proposed integral is reduced to one more simple, and this again, by the repetition of the process, to one yet more simple, and so on, till it is made to depend upon the simplest of its class.

The first form of u to which this method applies, and which demands particular attention on account of its frequent occurrence in the applications of the Integral Calculus, is the binomial differential coefficient

$$x^{m-1}(a+bx^n)^p, \text{ or } x^{m-1}X^p,$$

as we shall write it, m, n, p, being any numbers whatever, whole or fractional, positive or negative. Of course

$$x^{m-1}(ax^r+bx^n)^p$$

is included in this, since it may be written

$$x^{m+pr-1}(a+bx^{n-r})^p.$$

52. It has already been shewn, Art. 19, that when $\dfrac{m}{n}$ is a positive integer, or $\dfrac{m}{n}+p$ a negative integer, $\int_x x^{m-1}(a+bx^n)^p$ can be reduced to the sum of a series of terms of the form $c\int_z z^r$; and when either of these conditions is satisfied, this is the easiest method of integration, though it does not usually give the result in its simplest form. Also in the cases where $\dfrac{m}{n}$ is a negative integer, or $\dfrac{m}{n}+p$ a positive integer, the same substitutions

will transform the integral into $\int_z R(z)$, where $R(z)$ is a rational but not an integral function of z; consequently the method of substitution is of no practical use in the latter cases and must be superseded by the one which we are about to explain, founded upon the following proposition.

53. It is always possible to make an integral of the form $\int_x x^{m-1} X^p$ depend upon another of the same form, in which one of the indices is altered; viz. that of x by the addition or subtraction of n, and that of X by the addition or subtraction of unity.

For if we differentiate the expression $x^r X^q$, by the formula

$$d_x(u^m v^n) = u^{m-1} v^{n-1} \{ m v d_x u + n u d_x v \},$$

we find, since $\quad d_x X = n b x^{n-1}$,

$$d_x(x^r X^q) = x^{r-1} X^{q-1} (rX + qx \,.\, nbx^{n-1});$$

now, first, eliminate X from the quantity within brackets, by putting for it its value, $a + bx^n$,

$$\begin{aligned} \therefore\ d_x(x^r X^q) &= x^{r-1} X^{q-1} \{ ra + (r + nq) bx^n \} \\ &= rax^{r-1} X^{q-1} + (r + nq) bx^{r+n-1} X^{q-1}; \qquad (1) \end{aligned}$$

secondly, eliminate x^n from the quantity within brackets, by putting for bx^n its value, $X - a$,

$$\begin{aligned} \therefore\ d_x(x^r X^q) &= x^{r-1} X^{q-1} \{ (r + nq) X - anq \} \\ &= (r + nq) x^{r-1} X^q - nqax^{r-1} X^{q-1}. \qquad (2) \end{aligned}$$

Upon integrating equations (1) and (2), we find the integrals $\int_x x^{r-1} X^{q-1}$ and $\int_x x^{r+n-1} X^{q-1}$ connected, and $\int_x x^{r-1} X^q$ and $\int_x x^{r-1} X^{q-1}$ connected, (so that in each case, one is known if the other is) where the respective indices differ in the manner announced.

54. In the preceding article, we observe that the quantity differentiated, $x^r X^q$, is formed by adding unity to each index in the one of *lower* dimensions of the two expressions whose integrals are connected; and that in resolving its differential

coefficient, we eliminate X, or x^n, from the quantity within brackets, according as it is the index of X or x which is to remain unaltered.

55. Hence an integral of the form $\int_x x^{m-1}X^p$ being proposed, we must first consider whether, in order to reduce it to a known form, it must be made to depend upon an integral in which the index of x is changed, or upon one in which that of X is changed; i. e. with which of the four integrals

$$\int_x x^{m-n-1}X^p, \quad \int_x x^{m+n-1}X^p, \quad \int_x x^{m-1}X^{p-1}, \quad \int_x x^{m-1}X^{p+1},$$

it must be connected; and then make an assumption according to this rule.

Take the one of lower *dimensions of the two expressions whose integrals are to be connected, increase the index both of* x *and* X *by unity, and assume the result* = P.

Then in all cases d_xP can be resolved, (by performing, as the case requires, one or other of the eliminations to which the attention was called in the last article) into the sum of the two expressions whose integrals are to be connected, each multiplied by a constant coefficient; and upon integrating, the formula of reduction is obtained, by the successive applications of which, $\int_x x^{m-1}X^p$ is made to depend either upon a known form, or upon the simplest integral of its class.

Ex. 1. $$\int_x \frac{x^m}{\sqrt{a^2 - x^2}}.$$

Here we must diminish the index of x, i. e. connect the integral with $\int_x x^{m-2}(a^2 - x^2)^{-\frac{1}{2}}$; for since m will be diminished by 2 by each process, we shall at length arrive at

$$\int_x \frac{1}{\sqrt{a^2 - x^2}}, \text{ or } \int_x \frac{x}{\sqrt{a^2 - x^2}},$$

(according as m is even or odd) both of which are known forms.

Also since $x^{m-2}(a^2-x^2)^{-\frac{1}{2}}$ is the expression of lower dimensions, increasing each index by unity, the assumption is

$$P = x^{m-1}\sqrt{a^2 - x^2},$$

$$\therefore\ d_x P = \frac{x^{m-2}}{\sqrt{a^2 - x^2}}\left\{(m-1)(a^2 - x^2) - x^2\right\}$$

$$= \frac{x^{m-2}}{\sqrt{a^2 - x^2}}\left\{(m-1)a^2 - mx^2\right\}$$

$$= (m-1)a^2\frac{x^{m-2}}{\sqrt{a^2 - x^2}} - m\frac{x^m}{\sqrt{a^2 - x^2}};$$

$$\therefore\ \int_x \frac{x^m}{\sqrt{a^2 - x^2}} = -\frac{x^{m-1}X^{\frac{1}{2}}}{m} + \frac{m-1}{m}a^2\int_x \frac{x^{m-2}}{\sqrt{a^2 - x^2}},$$

which is the formula of reduction.

Change m into $m - 2$, $m - 4$, &c.

$$\therefore\ \int_x \frac{x^{m-2}}{\sqrt{a^2 - x^2}} = -\frac{x^{m-3}X^{\frac{1}{2}}}{m-2} + \frac{m-3}{m-2}a^2\int_x \frac{x^{m-4}}{\sqrt{a^2 - x^2}},$$

$$\int_x \frac{x^{m-4}}{\sqrt{a^2 - x^2}} = -\frac{x^{m-5}X^{\frac{1}{2}}}{m-4} + \frac{m-5}{m-4}a^2\int_x \frac{x^{m-6}}{\sqrt{a^2 - x^2}};$$

and we shall at last come either to

$$\int_x \frac{x^2}{\sqrt{a^2 - x^2}} = -\tfrac{1}{2}xX^{\frac{1}{2}} + \frac{a^2}{2}\int_x \frac{1}{\sqrt{a^2 - x^2}} = -\tfrac{1}{2}xX^{\frac{1}{2}} + \frac{a^2}{2}\sin^{-1}\frac{x}{a};$$

$$\text{or to } \int_x \frac{x^3}{\sqrt{a^2 - x^2}} = -\tfrac{1}{3}x^2X^{\frac{1}{2}} + \frac{2a^2}{3}\int_x \frac{x}{\sqrt{a^2 - x^2}}$$

$$= -\tfrac{1}{3}x^2X^{\frac{1}{2}} - \frac{2a^2}{3}\sqrt{a^2 - x^2}.$$

Hence, collecting the results, we have, when m is even,

$$\int_x \frac{x^m}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2}\left\{\frac{x^{m-1}}{m} + \frac{m-1}{m}\frac{a^2 x^{m-3}}{m-2}\right.$$

$$\left. + \frac{(m-1)(m-3)}{m(m-2)}\frac{a^4 x^{m-5}}{m-4} + \&\text{c.} + \frac{(m-1)(m-3)\ldots 1}{m(m-1)\ldots 2}a^{m-2}x\right\}$$

$$+ \frac{(m-1)(m-3)\ldots 1}{m(m-2)\ldots 4.2}a^m \sin^{-1}\frac{x}{a};$$

but if m be odd, $\displaystyle\int_x \frac{x^m}{\sqrt{a^2 - x^2}} = -\sqrt{a^2-x^2}\left\{\frac{x^{m-1}}{m} + \frac{m-1}{m}\frac{a^2x^{m-3}}{m-2}\right.$

$$\left. + \frac{(m-1)(m-3)}{m(m-2)}\frac{a^4x^{m-5}}{m-4} + \&\text{c.} + \frac{(m-1)(m-3)\ldots 2}{m(m-2)\ldots 3}a^{m-1}\right\}.$$

These results are homogeneous and of m dimensions in x and a, as they ought to be; for the expression integrated is homogeneous and of $m - 1$ dimensions.

If m be odd, then $\frac{m+1}{2}$ is a positive integer; but the integral obtained by substitution, as in Art. 19, would be in powers of $(a^2 - x^2)$.

Ex. 2. $$\int_x \frac{x^2}{\sqrt{2ax - x^2}}.$$

When as here, the index to be reduced is a given number, we must investigate the formula of reduction for the case where some general symbol m is written for the number, and then substitute.

In the present instance then we must find the formula of reduction for $\displaystyle\int_x \frac{x^m}{\sqrt{2ax - x^2}} = \int_x x^{m-\frac{1}{2}}(2a - x)^{-\frac{1}{2}}$, which must be connected with $\int_x x^{m-\frac{3}{2}}(2a - x)^{-\frac{1}{2}}$. Hence the assumption is

$$P = x^{m-\frac{1}{2}}(2a - x)^{\frac{1}{2}};$$

$$\therefore\ d_x P = \frac{x^{m-\frac{3}{2}}}{\sqrt{2a-x}} \left\{(m-\tfrac{1}{2})(2a-x) - \tfrac{1}{2}x\right\}$$

$$= \frac{x^{m-\frac{3}{2}}}{\sqrt{2a-x}} \left\{(m-\tfrac{1}{2})\,2a - mx\right\}$$

$$= (2m-1)a\,\frac{x^{m-1}}{\sqrt{2ax-x^2}} - m\,\frac{x^m}{\sqrt{2ax-x^2}};$$

$$\therefore\ \int_x \frac{x^m}{\sqrt{2ax-x^2}} = -\frac{x^{m-1}\sqrt{2ax-x^2}}{m} + \frac{(2m-1)\,a}{m}\int_x \frac{x^{m-1}}{\sqrt{2ax-x^2}}.$$

Hence making successively $m = 2$, $m = 1$,

$$\int_x \frac{x^2}{\sqrt{2ax-x^2}} = -\frac{x\sqrt{2ax-x^2}}{2} + \frac{3a}{2}\int_x \frac{x}{\sqrt{2ax-x^2}},$$

$$\int_x \frac{x}{\sqrt{2ax-x^2}} = -\sqrt{2ax-x^2} + a\int_x \frac{1}{\sqrt{2ax-x^2}}$$

$$= -\sqrt{2ax-x^2} + a\ \text{versin}^{-1}\frac{x}{a};$$

$$\therefore\ \int_x \frac{x^2}{\sqrt{2ax-x^2}} = -\sqrt{2ax-x^2}\left(\frac{x}{2} + \frac{3a}{2}\right) + \frac{3a^2}{2}\ \text{versin}^{-1}\frac{x}{a}.$$

Ex. 3. $\displaystyle\int_x \frac{1}{x^m\sqrt{a^2+x^2}} = \int_x x^{-m}(a^2+x^2)^{-\frac{1}{2}};$

in this case the index of x must be increased, that is, the integral must be made to depend upon $\int_x x^{-m+2}(a^2+x^2)^{-\frac{1}{2}}$; as[illegible]e $\therefore\ P = x^{-m+1}\sqrt{a^2+x^2}$, and we find the formula of [illegible]ction

$$\int_x \frac{1}{x^m\sqrt{a^2+x^2}} = -\frac{1}{(m-1)\,a^2}\,\frac{\sqrt{a^2+x^2}}{x^{m-1}}$$

$$-\frac{m-2}{(m-1)\,a^2}\int_x \frac{1}{x^{m-2}\sqrt{a^2+x^2}};$$

by this formula, when m is odd, we arrive at $\int_x \frac{1}{x\sqrt{a^2+x^2}}$, a known form; when m is even, we have the integration completely effected by it.

In the latter case, $\frac{-m+1}{2} - \frac{1}{2} = -\frac{m}{2}$, a negative integer, and the expression might be integrated by Art. 19.

Hence, making successively $m=5$, $m=3$, $\int_x \frac{1}{x^5\sqrt{a^2+x^2}}$

$$= \sqrt{a^2+x^2}\left(-\frac{1}{4a^2x^4}+\frac{3}{8a^4x^2}\right)+\frac{3}{8a^5}\log_e\left(\frac{x}{\sqrt{a^2+x^2}+a}\right).$$

Ex. 4. $\int_x (a^2-x^2)^{\frac{n}{2}}$ (n being odd).

Here we must diminish the index of X, and therefore connect the integral with $\int_x x^0 (a^2-x^2)^{\frac{n}{2}-1}$, and so shall arrive at $\int_x \sqrt{a^2-x^2}$ which is a known form; assume therefore

$$P = x(a^2-x^2)^{\frac{n}{2}},$$

$$\therefore\ d_xP = (a^2-x^2)^{\frac{n}{2}-1}.\left\{(a^2-x^2)-nx^2\right\}$$

$$= (a^2-x^2)^{\frac{n}{2}-1}\left\{(a^2-x^2)+n(a^2-x^2)-na^2\right\}$$

$$= (n+1)(a^2-x^2)^{\frac{n}{2}} - na^2(a^2-x^2)^{\frac{n}{2}-1};$$

$$\therefore\ \int_x (a^2-x^2)^{\frac{n}{2}} = \frac{x(a^2-x^2)^{\frac{n}{2}}}{n+1} + \frac{na^2}{n+1}\int_x (a^2-x^2)^{\frac{n-2}{2}}.$$

Hence, making successively $n=5$, $n=3$,

$$\int_x (a^2-x^2)^{\frac{5}{2}} = \frac{1}{6}\,x(a^2-x^2)^{\frac{5}{2}} + \frac{5a^2}{6.4}\,x(a^2-x^2)^{\frac{3}{2}}$$

$$+\frac{5.3a^4}{6.4.2}\,x(a^2-x^2)^{\frac{1}{2}} + \frac{5.3.1}{6.4.2}\,a^6\sin^{-1}\frac{x}{a}.$$

Ex. 5. $$\int_x \frac{1}{(a^2+x^2)^n}.$$

In this case we must increase the index of X, that is, connect the integral with $\int_x (a^2+x^2)^{-n+1}$, and so shall arrive at $\int_x \frac{1}{a^2+x^2}$; therefore,

$$P = x\,(a^2+x^2)^{-n+1};$$

$$\therefore\; d_x P = (a^2+x^2)^{-n}\left\{(a^2+x^2) - 2\,(n-1)\,x^2\right\}$$

$$= (a^2+x^2)^{-n}\left\{(a^2+x^2) - 2\,(n-1)\,(a^2+x^2) + 2\,(n-1)\,a^2\right\}$$

$$= -\,(2n-3)\,(a^2+x^2)^{-n+1} + (2n-2)\,a^2\,(a^2+x^2)^{-n};$$

$$\therefore\; \int_x \frac{1}{(a^2+x^2)^n} = \frac{xX^{-n+1}}{(2n-2)\,a^2} + \frac{2n-3}{2n-2}\cdot\frac{1}{a^2}\int_x \frac{1}{(a^2+x^2)^{n-1}};$$

therefore, changing n successively into $n-1$, $n-2$, &c.

$$\int_x \frac{1}{(a^2+x^2)^{n-1}} = \frac{xX^{-n+2}}{(2n-4)\,a^2} + \frac{2n-5}{2n-4}\cdot\frac{1}{a^2}\int_x \frac{1}{(a^2+x^2)^{n-2}},$$

$$\int_x \frac{1}{(a^2+x^2)^{n-2}} = \frac{xX^{-n+3}}{(2n-6)\,a^2} + \frac{2n-7}{2n-6}\cdot\frac{1}{a^2}\int_x \frac{1}{(a^2+x^2)^{n-3}};$$

till at last we come to

$$\int_x \frac{1}{(a^2+x^2)^2} = \frac{xX^{-1}}{2\,a^2} + \frac{1}{2a^2}\int_x \frac{1}{a^2+x^2} = \frac{xX^{-1}}{2a^2} + \frac{1}{2\,a^3}\tan^{-1}\frac{x}{a};$$

$$\int_x \frac{1}{(a^2+x^2)^n} = \frac{xX^{-n+1}}{(2n-2)\,a^2} + \frac{2n-3}{2n-2}\cdot\frac{xX^{-n+2}}{(2n-4)\,a^4}$$

$$+\frac{(2n-3)\,(2n-5)}{(2n-2)\,(2n-4)}\cdot\frac{xX^{-n+3}}{(2n-6)\,a^6} + \&\text{c.} + \frac{(2n-3)\,(2n-5)\ldots 3}{(2n-2)\,(2n-4)\ldots 4}\cdot\frac{xX^{-1}}{2a^{2n-2}}$$

$$+\frac{(2n-3)\,(2n-5)\ldots 3\,.\,1}{(2n-2)\,(2n-4)\ldots 4\,.\,2}\;\frac{1}{a^{2n-1}}\tan^{-1}\frac{x}{a}.$$

We may reduce the more general integral $\int_x \frac{1}{(a+bx+cx^2)^n}$ to the above form; for

$$\int_x \frac{1}{(a+bx+cx^2)^n} = \frac{1}{c^n}\int_x \frac{1}{\left(\frac{a}{c}+\frac{b}{c}x+x^2\right)^n} = \frac{1}{c^n}\int_z \frac{1}{(a^2+z^2)^n},$$

$$\text{making } x+\frac{b}{2c}=z, \quad \text{and } a^2=\frac{a}{c}-\frac{b^2}{4c^2}=\frac{4ac-b^2}{4c^2}.$$

$$\text{Also } \int_x \frac{1}{(f+gx)^n\sqrt{a+bx}} = \frac{2b^{n-1}}{g^n}\int_z \frac{1}{(a^2+z^2)^n},$$

$$\text{making } z=\sqrt{a+bx}, \quad \text{and } a^2=\frac{fb-ag}{g}.$$

Ex. 6. To make

$\int_x x^{m-1}(a+bx^n)^p$ depend upon $\int_x x^{m-n-1}(a+bx^n)^p$.

Assume $P = x^{m-n}X^{p+1}$,

$$\therefore\ d_xP = x^{m-n-1}X^p\left\{(m-n)X+(p+1)x.nbx^{n-1}\right\}$$
$$= x^{m-n-1}X^p\left\{(m-n)(a+bx^n)+(np+n)bx^n\right\}$$
$$= (m-n)ax^{m-n-1}X^p+(m+np)bx^{m-1}X^p;$$

$$\therefore\ \int_x x^{m-1}X^p = \frac{x^{m-n}X^{p+1}}{(m+np)b} - \frac{(m-n)a}{(m+np)b}\int_x x^{m-n-1}X^p.$$

Similarly, to make

$\int_x x^{m-1}(a+bx^n)^p$ depend upon $\int_x x^{m+n-1}(a+bx^n)^p$,

the assumption is $P = x^mX^{p+1}$, and the result

$$\int_x x^{m-1}X^p = \frac{x^mX^{p+1}}{ma} - \frac{(m+n+np)b}{ma}\int_x x^{m+n-1}X^p.$$

Ex. 7. To make

$\int_x x^{m-1}(a+bx^n)^p$ depend upon $\int_x x^{m-1}(a+bx^n)^{p-1}$.

Assume $P = x^m X^p$,

$$\therefore\ d_x P = x^{m-1}X^{p-1}\{mX + px \, . \, nbx^{n-1}\}$$

$$= x^{m-1}X^{p-1}\{mX + np \, . \, (X-a)\}$$

$$= (m+np)\, x^{m-1}X^p - npax^{m-1}X^{p-1};$$

$$\therefore\ \int_x x^{m-1}X^p = \frac{x^m X^p}{m+np} + \frac{npa}{m+np}\int_x x^{m-1}X^{p-1}.$$

Similarly, to make

$\int_x x^{m-1}(a+bx^n)^p$ depend upon $\int_x x^{m-1}(a+bx^n)^{p+1}$,

the assumption is $P = x^m X^{p+1}$, and the result

$$\int_x x^{m-1}X^p = -\frac{x^m X^{p+1}}{(p+1)\,na} + \frac{m+n+np}{(p+1)\,na}\int_x x^{m-1}X^{p+1}.$$

56. This method fails when any of the coefficients become infinite. By inspecting the results in examples 6 and 7 of the preceding article, we see that this will happen (1) when $p=-1$, or the index of X is -1, (2) when $m=0$, or the index of x is -1, and (3) when $m+np=0$, or $p=-\frac{m}{n}$.

The first case of failure is $\int_x \frac{x^{m-1}}{a+bx^n}$, a rational fraction already integrated.

The second is $\int_x \frac{1}{x}(a+bx^n)^{\frac{p}{q}}$, (changing p into $\frac{p}{q}$, as the
or it is to be immediately intg^d q
index must be fractional) which, by making $a+bx^n = z^q$, becomes

$$\frac{q}{n}\int_z \frac{z^{p+q-1}}{z^q-a}.$$

$a + bx^n = z^q$

$\therefore\ x^n = \frac{z^q - a}{b}$

$\frac{n}{x} = \frac{q z^{q-1} \frac{dz}{dx}}{b(z^q - a)}$

$\therefore\ \frac{1}{x}(a+bx^n) = \frac{q}{n}\left(\frac{z^{q-1}\frac{dz}{dx}}{z^q - a}\right)$

The third is $\int_x \frac{x^{m-1}}{(a+bx^n)^{\frac{m}{n}}} = \int_x \frac{1}{x\,(ax^{-n}+b)^{\frac{m}{n}}}$,

the same as the second; make therefore $ax^{-n} + b = z^n$.

Ex. 1. $\int_x \frac{1}{(x^4-1)^{\frac{1}{4}}} = \int_x \frac{1}{x\,(1-x^{-4})^{\frac{1}{4}}}$.

Make $1 - x^{-4} = z^4$,

$$\therefore \int_x \frac{1}{(x^4-1)^{\frac{1}{4}}} = \int_z \frac{z^2}{1-z^4}$$

$$= \tfrac{1}{2} \int_z \left(\frac{1}{1-z^2} - \frac{1}{1+z^2}\right) = \tfrac{1}{4} \log_\epsilon \frac{1+z}{1-z} - \tfrac{1}{2} \tan^{-1} z.$$

Ex. 2. $$\int_x \frac{1}{x\,(a+bx)^{\frac{2}{3}}}$$

$$= \frac{3}{2a^{\frac{2}{3}}} \log_\epsilon \frac{(a+bx)^{\frac{1}{3}} - a^{\frac{1}{3}}}{x^{\frac{1}{3}}} - \frac{\sqrt{3}}{a^{\frac{2}{3}}} \tan^{-1} \frac{2\,(a+bx)^{\frac{1}{3}} + a^{\frac{1}{3}}}{\sqrt{3}\; a^{\frac{1}{3}}}.$$

57. We can also make $\int_x x^{m-1}(a+bx^n)^p$ depend upon an integral of the same form in which both the indices are altered; namely, that of x by the addition or subtraction of n, and that of X by the subtraction or addition of unity. For this, the simple application of the formula for integration by parts is sufficient.

Thus, commencing the integration with the factor x^{m-1},

$$\int_x x^{m-1}(a+bx^n)^p = \frac{x^m}{m} \cdot (a+bx^n)^p - \frac{npb}{m} \int_x x^{m+n-1}(a+bx^n)^{p-1}.$$

When we commence the integration with the other factor, it will be necessary to prepare the expression by writing it

$$x^{m-n} \,.\, x^{n-1}(a+bx^n)^p;$$

Rule "When y^e index of x is to be increased commence with integ^o x^{m-1}; but when that of X commence with integ^o X^{p}."

$$\therefore \int_x x^{m-1}(a+bx^n)^p$$

$$= x^{m-n}\cdot\frac{(a+bx^n)^{p+1}}{(p+1)nb} - \frac{m-n}{(p+1)nb}\int_x x^{m-n-1}(a+bx^n)^{p+1}.$$

We observe that these formulæ fail when $m=0$, or $p=-1$, both which cases have already been considered.

Ex. 1. $$\int_x \frac{(a+bx^2)^{n+\frac{1}{2}}}{x^m}.$$

This must be connected with $\int_x \frac{(a+bx^2)^{n-\frac{1}{2}}}{x^{m-2}}$;

therefore, commencing the integration with the factor $\frac{1}{x^m}$,

$$\int_x \frac{(a+bx^2)^{n+\frac{1}{2}}}{x^m} = -\frac{(a+bx^2)^{n+\frac{1}{2}}}{(m-1)\,x^{m-1}} + \frac{2n+1}{m-1}\,b\int_x \frac{(a+bx^2)^{n-\frac{1}{2}}}{x^{m-2}}.$$

Hence $$\int_x \frac{(a+bx^2)^{\frac{3}{2}}}{x^3} = -\frac{(a+bx^2)^{\frac{3}{2}}}{2x^2} + \frac{3b}{2}\int_x \frac{(a+bx^2)^{\frac{1}{2}}}{x};$$

and an integral similar to the last has been already found p. 30.

Ex. 2. $$\int_x \frac{x^m}{(1+x^2)^n}.$$

This must be connected with $\int_x \frac{x^{m-2}}{(1+x^2)^{n-1}}$; therefore we must commence the integration with the factor $\frac{1}{(1+x^2)^n}$, and consequently write the expression, $x^{m-1}\cdot\frac{x}{(1+x^2)^n}$;

$$\therefore \int_x \frac{x^m}{(1+x^2)^n} = -\frac{x^{m-1}}{(2n-2)(1+x^2)^{n-1}} + \frac{m-1}{2n-2}\int_x \frac{x^{m-2}}{(1+x^2)^{n-1}}.$$

Hence making $m = 4$, $n = 5$, &c.,

$$\int_x \frac{x^4}{(1+x^2)^5} = -\frac{1}{8}\frac{x^3}{(1+x^2)^4} + \frac{3}{8}\int_x \frac{x^2}{(1+x^2)^4},$$

$$\int_x \frac{x^2}{(1+x^2)^4} = -\frac{1}{6}\frac{x}{(1+x^2)^3} + \frac{1}{6}\int_x \frac{1}{(1+x^2)^3};$$

and $\displaystyle\int_x \frac{1}{(1+x^2)^3} = \frac{1}{4}\frac{x}{(1+x^2)^2} + \frac{3}{8}\frac{x}{1+x^2} + \frac{3}{8}\tan^{-1} x,$

by Ex. 5. Art. 55.

Hence, collecting the results, $\displaystyle\int_x \frac{x^4}{(1+x^2)^5}$ is obtained.

58. The method of assumptions is also applicable to the more general expression

$$\textstyle\int_x x^{m-1}(a + bx^n + cx^{2n})^p,$$

but in this case the integrals of three similar functions are connected.

For, putting $a + bx^n + cx^{2n} = X$, we have $d_x(x^r X^q)$

$$= x^{r-1}X^{q-1}\{rX + qx(nbx^{n-1} + 2ncx^{2n-1})\}$$

$$= x^{r-1}X^{q-1}\{r(a + bx^n + cx^{2n}) + q(nbx^n + 2ncx^{2n})\}$$

$$= rax^{r-1}X^{q-1} + (r + nq)bx^{r+n-1}X^{q-1} + (r + 2nq)cx^{r+2n-1}X^{q-1};$$

hence, upon integrating, we have an integral of the form $\int_x x^{m-1}X^p$ connected with two others in which the index of x is increased or diminished, (that of X remaining unchanged) according as we consider $\int_x x^{r-1}X^{q-1}$ or $\int_x x^{r+2n-1}X^{q-1}$ to be the proposed integral.

Ex. 1. To make $\int_x x^{m-1}X^p$ depend upon $\int_x x^{m-n-1}X^p$ and $\int_x x^{m-2n-1}X^p$.

Assume $P = x^{m-2n} X^{p+1}$,

$x^{m-2n-1}X^p$ being the one of lowest dimensions of the functions whose integrals are to be connected;

$$\therefore\ d_x P = x^{m-2n-1}X^p\left\{(m-2n)X+(p+1)x(nbx^{n-1}+2ncx^{2n-1})\right\},$$

$$= x^{m-2n-1}X^p\left\{(m-2n)(a+bx^n+cx^{2n})+(np+n)(bx^n+2cx^{2n})\right\},$$

$$=(m-2n)\,ax^{m-2n-1}X^p+(m-n+np)\,bx^{m-n-1}X^p$$
$$+(m+2np)\,cx^{m-1}X^p;$$

therefore, integrating, $\int_x x^{m-1}X^p = \dfrac{x^{m-2n}X^{p+1}}{(m+2np)c}$

$$-\frac{(m-2n)a}{(m+2np)c}\int_x x^{m-2n-1}X^p-\frac{(m-n+np)b}{(m+2np)c}\int_x x^{m-n-1}X^p.$$

Ex. 2. To make $\int_x x^{m-1}X^p$ depend upon $\int_x x^{m+n-1}X^p$ and $\int_x x^{m+2n-1}X^p$.

Assume $P = x^m X^{p+1}$, and the formula of the reduction is

$$\int_x x^{m-1}X^p = \frac{x^m X^{p+1}}{ma}-\frac{(m+n+np)b}{ma}\int_x x^{m+n-1}X^p$$

$$-\frac{(m+2n+2np)c}{ma}\int_x x^{m+2n-1}X^p.$$

The formulæ of the two preceding examples will always enable us to reduce $\int_x x^m(a+bx+cx^2)^p$ to $\int_x(a+bx+cx^2)^p$, when m is a positive or negative integer.

Ex. 3. $$\int_x\frac{x^m}{\sqrt{a+bx+cx^2}};$$

this may be made to depend upon $\displaystyle\int_x\frac{x^{m-1}}{\sqrt{X}}$ and $\displaystyle\int_x\frac{x^{m-2}}{\sqrt{X}}$, by assuming $P = x^{m-1}\sqrt{X}$, and the formula of reduction is

$$\int_x \frac{x^m}{\sqrt{X}} = \frac{x^{m-1}\sqrt{X}}{mc} - \frac{m-1}{m}\,\frac{a}{c}\int_x \frac{x^{m-2}}{\sqrt{X}} - \frac{2m-1}{2m}\,\frac{b}{c}\int_x \frac{x^{m-1}}{\sqrt{X}}.$$

Let $m = 2$; $\therefore$ $\displaystyle\int_x \frac{x^2}{\sqrt{X}} = \frac{x\sqrt{X}}{2c} - \frac{a}{2c}\int_x \frac{1}{\sqrt{X}} - \frac{3b}{4c}\int_x \frac{x}{\sqrt{X}}$

$$= \frac{x\sqrt{X}}{2c} - \frac{a}{2c}\int_x \frac{1}{\sqrt{X}} - \frac{3b}{4c^2}\int_x \frac{2cx + b - b}{2\sqrt{X}}$$

$$= \frac{x\sqrt{X}}{2c} - \frac{3b}{4c^2}\sqrt{X} - \left(\frac{a}{2c} - \frac{3b^2}{8c^2}\right)\int_x \frac{1}{\sqrt{X}}$$

$$= \frac{2cx - 3b}{4c^2}\sqrt{X} + \frac{4ac - 3b^2}{8c^2\sqrt{c}}\log_e\left(2cx + b + 2\sqrt{cX}\right).$$

Ex. 4. $\displaystyle\int_x \frac{1}{x^m\sqrt{a + bx + cx^2}}$;

to make this depend upon $\displaystyle\int_x \frac{1}{x^{m-1}\sqrt{X}}$, and $\displaystyle\int_x \frac{1}{x^{m-2}\sqrt{X}}$, assume $P = x^{-m+1}\sqrt{X}$, and the formula of reduction is

$$\int_x \frac{1}{x^m\sqrt{X}} = -\frac{\sqrt{X}}{(m-1)\,ax^{m-1}} - \frac{2m-3}{2m-2}\,\frac{b}{a}\int_x \frac{1}{x^{m-1}\sqrt{X}}$$
$$- \frac{m-2}{m-1}\,\frac{c}{a}\int_x \frac{1}{x^{m-2}\sqrt{X}}.$$

Make $m = 3$, 2, successively, and collect the results,

$$\therefore \int_x \frac{1}{x^3\sqrt{X}} = -\frac{\sqrt{X}}{2ax^2} + \frac{3b\sqrt{X}}{4a^2x} + \left(\frac{3b^2}{8a^2} - \frac{c}{2a}\right)\int_x \frac{1}{x\sqrt{X}}.$$

For the last integral, see Ex. 4. Art. 27.

Ex. 5. $\int_x \frac{x^m}{(a+bx+cx^2)^{\frac{3}{2}}}$;

to make this depend upon $\int_x \frac{x^{m-1}}{X^{\frac{3}{2}}}$, and $\int_x \frac{x^{m-2}}{X^{\frac{3}{2}}}$, assume $P = x^{m-1}X^{-\frac{1}{2}}$, and the formula of reduction is

$$\int_x \frac{x^m}{X^{\frac{3}{2}}} = \frac{x^{m-1}}{(m-2)c\sqrt{X}} - \frac{m-1}{m-2}\frac{a}{c}\int_x \frac{x^{m-2}}{X^{\frac{3}{2}}} - \frac{2m-3}{2m-4}\frac{b}{c}\int_x \frac{x^{m-1}}{X^{\frac{3}{2}}}.$$

Let $m = 3$; $\therefore \int_x \frac{x^3}{X^{\frac{3}{2}}} = \frac{x^2}{c\sqrt{X}} - \frac{2a}{c}\int_x \frac{x}{X^{\frac{3}{2}}} - \frac{3b}{2c}\int_x \frac{x^2}{X^{\frac{3}{2}}}.$

But $\int_x \frac{x^2}{X^{\frac{3}{2}}} = \frac{1}{c}\int_x \frac{X-(a+bx)}{X^{\frac{3}{2}}} = \frac{1}{c}\int_x \frac{1}{\sqrt{X}} - \frac{a}{c}\int_x \frac{1}{X^{\frac{3}{2}}} - \frac{b}{c}\int_x \frac{x}{X^{\frac{3}{2}}}$;

$$\therefore \int_x \frac{x^3}{X^{\frac{3}{2}}} = \frac{x^2}{c\sqrt{X}} - \left(\frac{2a}{c} - \frac{3b^2}{2c^2}\right)\int_x \frac{x}{X^{\frac{3}{2}}} - \frac{3b}{2c^2}\int_x \frac{1}{\sqrt{X}} + \frac{3ab}{2c^2}\int_x \frac{1}{X^{\frac{3}{2}}};$$

but $\int_x \frac{x}{X^{\frac{3}{2}}} = \int_x \frac{x^{-2}}{(ax^{-2}+bx^{-1}+c)^{\frac{3}{2}}}$

$$= -4\sqrt{a}\int_x \frac{d_x(2ax^{-1}+b)}{\{(2ax^{-1}+b)^2+4ac-b^2\}^{\frac{3}{2}}} = -\frac{2(2a+bx)}{k\sqrt{X}},$$

(Ex. 1. Art. 13.) making $4ac - b^2 = k$; also

$$\int_x \frac{1}{X^{\frac{3}{2}}} = 4\sqrt{c}\int_x \frac{d_x(2cx+b)}{\{(2cx+b)^2+4ac-b^2\}^{\frac{3}{2}}} = \frac{2(2cx+b)}{k\sqrt{X}};$$

$$\therefore \int_x \frac{x^3}{X^{\frac{3}{2}}} = \frac{x^2}{c\sqrt{X}} + \frac{4ac-3b^2}{c^2k}\cdot\frac{2a+bx}{\sqrt{X}} + \frac{3ab}{kc^2}\frac{2cx+b}{\sqrt{X}}$$

$$-\frac{3b}{2c^2}\int_x \frac{1}{\sqrt{X}}.$$

59. We can also make $\int_x x^{m-1}(a+bx^n+cx^{2n})^p$ depend upon integrals in which both the indices are altered, viz., that of X diminished, and that of x increased, by integrating by parts, which gives

$$\int_x x^{m-1}X^p = \frac{x^m X^p}{m} - \frac{1}{m}\int_x x^m p X^{p-1}(nbx^{n-1}+2ncx^{2n-1})$$

$$= \frac{x^m X^p}{m} - \frac{pnb}{m}\int_x x^{m+n-1}X^{p-1} - \frac{2pnc}{m}\int_x x^{m+2n-1}X^{p-1};$$

or, since the integral in the second member

$$\frac{np}{m}\int_x x^m X^{p-1}(bx^{n-1}+2cx^{2n-1}) = \frac{np}{m}\int_x x^{m-1}X^{p-1}(2X-2a-bx^n),$$

the above formula may be replaced by

$$\int_x x^{m-1}X^p = \frac{x^m X^p}{m+2np} + \frac{2pna}{m+2np}\int_x x^{m-1}X^{p-1}$$

$$+ \frac{pnb}{m+2np}\int_x x^{m+n-1}X^{p-1}.$$

60. In the case where the index of X is to be increased, if we assume

$$\int_x x^{m-1}X^p = (Ax^m + Bx^{m+n})X^{p+1} + \int_x (Cx^{m-1} + Dx^{m+n-1})X^{p+1},$$

differentiate, and divide by $x^{m-1}X^p$, the result will be of the form $0 = \alpha + \beta x^n + \gamma x^{2n} + \delta x^{3n}$; hence, equating to 0 the coefficients of the powers of x^n, we shall have four simple equations for finding the four constants A, B, C, D; their values will be found to be fractions whose numerators are respectively

$2ac-b^2$, $-bc$, $(np+n)(b^2-4ac)-m(2ac-b^2)$, $(2pn+3n+m)bc$,

and common denominator $(np+n)(b^2-4ac)a$.

Ex. $$\int_x \frac{1}{(a+bx^2+cx^4)^p};$$

assuming $\int_x \frac{1}{X^p} = \frac{Ax + Bx^3}{X^{p-1}} + \int_x \frac{C + Dx^2}{X^{p-1}}$, differentiating, and equating coefficients of like powers of x to determine the constants, we have, making $k = b^2 - 4ac$,

$$\int_x \frac{1}{X^p} = \frac{bcx^3 + (b^2 - 2ac)x}{2ak(p-1)X^{p-1}} + \frac{(4p-5)bc}{2ak(p-1)} \int_x \frac{x^2}{X^{p-1}}$$

$$+ \frac{2k(p-1) + 2ac - b^2}{2ak(p-1)} \int_x \frac{1}{X^{p-1}}.$$

By this formula (when p is a positive integer), $\int_x \frac{1}{X^p}$ can be reduced to an integral of the form

$$\int_x (a_0 + a_1 x^2 + a_2 x^4 + \dots + a_{p-1} x^{2(p-1)}) \frac{1}{X},$$

and, therefore, falls under Art. 48.

61. It may be shewn, exactly as in Art. 58, that the general integral

$$\int_x x^{m-1} (a + bx^n + cx^{2n} + \dots + tx^{kn})^p \text{ or } \int_x x^{m-1} X^p$$

can be made to depend upon k others of the same form in which the index of x is increased or diminished by n, $2n$, &c. that of X remaining unchanged; the quantity to be differentiated in each case, $x^m X^{p+1}$, or $x^{m-kn} X^{p\pm1}$, being formed by taking the one of lowest dimensions of the expressions whose integrals are to be connected and increasing each index by unity. Also by integrating by parts as in Art. 59, we can simultaneously diminish the index of X and increase that of x. When the index of X is to be increased, we must assume

$$\int_x x^{m-1} X^p = (A + Bx^n + Cx^{2n} + \dots + Tx^{(k-1)n}) x^m X^{p+1}$$

$$+ \int_x (A' + B'x^n + C'x^{2n} + \dots + T'x^{(k-1)n}) x^{m-1} X^{p+1},$$

and proceeding as in the last Art. we shall have $2k$ simple equations to determine the $2k$ constants A, B, &c.

SECTION V.

EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

ART. 62. WE next come to the consideration of $\int_x u$, where u involves exponential functions of x; the number of cases in which this integration can be completely effected is very limited.

Since $d_x(a^u) = \log_\epsilon a\, a^u d_x u$, the fundamental formula is

$$\int_x a^u d_x u = \frac{a^u}{\log_\epsilon a} + C.$$

Hence $\int_x a^x = \dfrac{a^x}{\log_\epsilon a} + C$, and $\int_x e^x = e^x + C$.

63. If $u = f(a^x)$ be an algebraic function of a^x,

$$\text{making } a^x = z, \text{ and } \therefore d_z x = \frac{1}{z \log_\epsilon a},$$

$$\text{we have } \int_x f(a^x) = \frac{1}{\log_\epsilon a} \int_z \frac{f(z)}{z},$$

which is algebraic with respect to z.

Ex. 1. $$\int_x \frac{1}{\sqrt{1 + a^x}} = \frac{1}{\log_\epsilon a} \int_z \frac{1}{z\sqrt{1 + z}}$$

$$= \frac{2}{\log_\epsilon a} \int_z \frac{d_z \sqrt{z}}{\sqrt{z}\sqrt{1+z}} = \frac{2}{\log_\epsilon a} . \log_\epsilon \frac{\sqrt{z}}{\sqrt{1+z}+1}$$

$$= \frac{2}{\log_\epsilon a} . \log_\epsilon \left(\sqrt{a^{-x} + 1} - \sqrt{a^{-x}}\right).$$

Ex. 2. $$\int_x \frac{e^{2x} + 1}{e^{2x} - 1} = \log_\epsilon (e^x - e^{-x}) + C.$$

64. The expression $e^x u$ can be integrated, whenever u can be split into the sum of two quantities, one of which is the differential coefficient of the other, for it is manifest that

$$\int_x e^x (y + d_x y) = e^x y.$$

Ex. 1. $\displaystyle\int_x \frac{e^x x}{(1+x)^2} = \int_x e^x \left(\frac{1}{1+x} - \frac{1}{(1+x)^2}\right) = \frac{e^x}{1+x};$

Ex. 2. $\displaystyle\int_x \frac{e^x (2 - x^2)}{(1-x)\sqrt{1-x^2}} = \int_x e^x \sqrt{\frac{1-x}{1+x}} \cdot \frac{2-x^2}{(1-x)^2}$

$$= \int_x e^x \sqrt{\frac{1-x}{1+x}} \left\{\frac{1+x}{1-x} + \frac{1}{(1-x)^2}\right\} = e^x \sqrt{\frac{1+x}{1-x}}.$$

65. To integrate $a^x u$, u being an algebraic function of x.

Case I. Let u be whole and rational.

Integrating by parts, that is, making $U = u$, and $d_x V = a^x$, in the formula $\int_x U d_x V = UV - \int_x V d_x U$, we find

$$\int_x a^x u = \frac{a^x u}{\log_\epsilon a} - \frac{1}{\log_\epsilon a} \int_x (a^x d_x u),$$

$$\text{similarly,}\ \int_x (a^x d_x u) = \frac{a^x d_x u}{\log_\epsilon a} - \frac{1}{\log_\epsilon a} \int_x (a^x d_x^2 u), \text{ \&c.}$$

$$\therefore\ \int_x a^x u = \frac{a^x u}{\log_\epsilon a} - \frac{a^x d_x u}{(\log_\epsilon a)^2} + \frac{a^x d_x^2 u}{(\log_\epsilon a)^3} - \text{\&c.}$$

$$+ (-1)^{n-1} \frac{a^x d_x^{n-1} u}{(\log_\epsilon a)^n} + \frac{(-1)^n}{(\log_\epsilon a)^n} \int_x (a^x d_x^n u).$$

Hence, when u is an integral and rational function of n dimensions, since $d_x^n u$ is a constant quantity, $a^x u$ can be completely integrated.

Ex. $\int_x a^x x^n$.

Here $u = x^n$, $\therefore\ d_x u = nx^{n-1}$, &c., $d_x^n u = n(n-1)\ldots 3.2.1$;

$$\therefore\ \int_x a^x x^n = a^x \left(\frac{x^n}{\log_\epsilon a} - \frac{nx^{n-1}}{(\log_\epsilon a)^2} + \frac{n(n-1)x^{n-2}}{(\log_\epsilon a)^3} - \text{\&c.}\right.$$

$$\left. + (-1)^n . \frac{n(n-1\ldots 3.2.1}{(\log_\epsilon a)^{n+1}}\right).$$

66. Case II. Let u be fractional.

Change U into a^x, and $d_x V$ into u, and $\therefore$ V into $\int_x u$;

$$\therefore\ \int_x a^x u = a^x \int_x u - \log_\epsilon a \int_x (a^x \int_x u),$$

$$\int_x (a^x \int_x u) = a^x \int_x^2 u - \log_\epsilon a \int_x (a^x \int_x^2 u), \text{ \&c.},$$

$$\therefore\ \int_x a^x u = a^x \int_x u - \log_\epsilon a \,.\, a^x \int_x^2 u + (\log_\epsilon a)^2 \,.\, a^x \int_x^3 u - \text{\&c.}$$

$$+ (-1)^{n-1} (\log_\epsilon a)^{n-1} a^x \int_x^n u + (-1)^n (\log_\epsilon a)^n . \int_x (a^x \int_x^n u).$$

Ex. $\displaystyle\int_x \frac{a^x}{x^n}$;

$$\text{here } u = x^{-n},\ \int_x u = \frac{-x^{-n+1}}{n-1},$$

$$\int_x^2 u = \frac{x^{-n+2}}{(n-1)(n-2)}, \text{ \&c.}, \int_x^{n-1} u = (-1)^{n-1} . \frac{x^{-1}}{(n-1)(n-2)\ldots 2.1};$$

$$\therefore\ \int_x \frac{a^x}{x^n} = -\frac{a^x}{(n-1)x^{n-1}} - \frac{a^x \log_\epsilon a}{(n-1)(n-2)x^{n-2}} - \text{\&c.}$$

$$+ \frac{(\log_\epsilon a)^{n-1}}{(n-1)(n-2)\ldots 2.1} . \int_x \frac{a^x}{x} . \quad +$$

The sign is + in the last term since it includes $\int \frac{a^x}{x}$, whereas in [illegible] [illegible] term [illegible] [illegible] [illegible]

67. The integral of $\frac{a^x}{x}$ can only be obtained in a series, thus,

$$\int_x \frac{a^x}{x} = \int_x \frac{1}{x}\left(1 + \frac{x \log_\epsilon a}{1} + \frac{x^2 (\log_\epsilon a)^2}{1.2} + \&c.\right)$$

$$= \log_\epsilon x + \frac{x \log_\epsilon a}{1} + \frac{x^2 (\log_\epsilon a)^2}{1.2.2} + \frac{x^3 (\log_\epsilon a)^3}{1.2.3.3} + \&c.$$

68. We next come to the case of $\int_x u$, where u involves logarithmic functions of the variable.

Expressions of the form $u \log_\epsilon v$ (u and v being algebraic functions of x) may sometimes be integrated by parts; thus

$$\int_x u \log_\epsilon v = \log_\epsilon v \, . \int_x u - \int_x \frac{d_x v \, . \int_x u}{v} \, .$$

Ex. 1. $$\int_x \frac{x \log_\epsilon x}{\sqrt{a^2 + x^2}} = \log_\epsilon x \, . \sqrt{a^2 + x^2} - \int_x \frac{1}{x} \sqrt{a^2 + x^2},$$

$$= \log_\epsilon x \, . \sqrt{a^2 + x^2} - \sqrt{a^2 + x^2} - a \log_\epsilon \left(\frac{x}{a + \sqrt{a^2 + x^2}}\right),$$

$$= \log_\epsilon \left(\frac{x}{\epsilon}\right) . \sqrt{a^2 + x^2} - a \log_\epsilon \left(\frac{x}{a + \sqrt{a^2 + x^2}}\right).$$

Ex. 2. $$\int_x x \sqrt{a^2 + x^2} \, . \log_\epsilon \left(\frac{x^2 - b^2}{b^2}\right) = \tfrac{1}{3} (a^2 + x^2)^{\frac{3}{2}} \log_\epsilon \left(\frac{x^2 - b^2}{b^2}\right)$$

$$- \tfrac{2}{3} \left\{ c^2 \sqrt{a^2 + x^2} + \tfrac{1}{3} (a^2 + x^2)^{\frac{3}{2}} + c^3 \log_\epsilon \frac{\sqrt{x^2 - b^2}}{c + \sqrt{x^2 + a^2}} \right\}$$

(making $c^2 = a^2 + b^2$).

69. To integrate $u (\log_\epsilon x)^n$, u denoting an algebraic function of x.

Let $\int_x u = u_1$, $\int_x \frac{u_1}{x} = u_2$, $\int_x \frac{u_2}{x} = u_3$, &c.;

$$\therefore\ \int_x u\,(\log_\epsilon x)^n = (\log_\epsilon x)^n u_1 - n\int_x (\log_\epsilon x)^{n-1}.\,\frac{u_1}{x},$$

$$\int_x \frac{u_1}{x}\,(\log_\epsilon x)^{n-1} = (\log_\epsilon x)^{n-1} u_2 - (n-1)\int_x (\log_\epsilon x)^{n-2}.\,\frac{u_2}{x},\ \text{\&c.};$$

$$\therefore\ \int_x u\,(\log_\epsilon x)^n = (\log_\epsilon x)^n u_1 - n\,(\log_\epsilon x)^{n-1} u_2$$

$$+\,n\,(n-1)\,(\log_\epsilon x)^{n-2} u_3 - \text{\&c.} + (-1)^n n\,(n-1)\ldots 2.1\,u_{n+1}.$$

Ex. 1. $\int_x x^m\,(\log_\epsilon x)^n$;

in this case $u_1 = \dfrac{x^{m+1}}{m+1}$,

$$u_2 = \frac{x^{m+1}}{(m+1)^2},\quad u_3 = \frac{x^{m+1}}{(m+1)^3},\ \text{\&c.}$$

$$\therefore\ \int_x x^m\,(\log_\epsilon x)^n = \frac{x^{m+1}}{m+1}\left\{(\log_\epsilon x)^n - \frac{n}{m+1}\,(\log_\epsilon x)^{n-1}\right.$$

$$\left. + \frac{n\,(n-1)}{(m+1)^2}\,(\log_\epsilon x)^{n-2} - \text{\&c.} + (-1)^n.\,\frac{n\,(n-1)\ldots 3.2.1}{(m+1)^n}\right\}.$$

Ex. 2. $\int_x x^{m+nx}$;

this can be found only in an infinite series by expanding x^{nx}, and then falls under the present case.

$$\int_x x^m.\,x^{nx} = \int_x x^m\left\{1 + nx\log_\epsilon x\right.$$

$$\left. + \frac{n^2x^2\,(\log_\epsilon x)^2}{1.2} + \frac{n^3x^3\,(\log_\epsilon x)^3}{1.2.3} + \text{\&c.}\right\}$$

$$= \int_x x^m + n\int_x x^{m+1}\log_\epsilon x + \frac{n^2}{1.2}\int_x x^{m+2}(\log_\epsilon x)^2$$

$$+ \frac{n^3}{1.2.3}\int_x x^{m+3}\,(\log_\epsilon x)^3 + \text{\&c}$$

$$= \frac{x^{m+1}}{m+1} + \frac{nx^{m+2}}{m+2}\left(\log_\epsilon x - \frac{1}{m+2}\right)$$

$$+\frac{n^2}{1.2}\,\frac{x^{m+3}}{m+3}\left((\log_\epsilon x)^2-\frac{2}{m+3}\log_\epsilon x+\frac{2.1}{(m+3)^2}\right)$$

$$+\frac{n^3}{1.2.3}\,\frac{x^{m+4}}{m+4}\left\{(\log_\epsilon x)^3-\frac{3}{m+4}(\log_\epsilon x)^2\right.$$

$$\left.+\frac{3.2}{(m+4)^2}\log_\epsilon x-\frac{3.2.1}{(m+4)^3}\right\}+\text{\&c. (by the preceding example)}$$

$$=x^{m+1}\left\{\frac{1}{m+1}-\frac{nx}{(m+2)^2}+\frac{n^2x^2}{(m+3)^3}-\text{\&c.}\right\}$$

$$+\frac{nx^{m+2}\log_\epsilon x}{1}\left\{\frac{1}{m+2}-\frac{nx}{(m+3)^2}+\frac{n^2x^2}{(m+4)^3}-\text{\&c.}\right\}$$

$$+\frac{n^2x^{m+3}(\log_\epsilon x)^2}{1.2}\left\{\frac{1}{m+3}-\frac{nx}{(m+4)^2}+\frac{n^2x^2}{(m+5)^3}-\text{\&c.}\right\}+\text{\&c.}$$

70. To integrate $u\,(\log_\epsilon x)^{-n}$.

Let $d_x(ux)=u_1,\quad d_x(u_1x)=u_2,\quad d_x(u_2x)=u_3$, &c.;

$$\therefore\ \int_x u\,(\log_\epsilon x)^{-n}=\int_x ux\,.\,\frac{1}{x}(\log_\epsilon x)^{-n}$$

$$=-\frac{ux}{n-1}(\log_\epsilon x)^{-n+1}+\frac{1}{n-1}\int_x u_1\,(\log_\epsilon x)^{-n+1},$$

$$\int_x u_1(\log_\epsilon x)^{-n+1}=-\frac{u_1x}{n-2}(\log_\epsilon x)^{-n+2}+\frac{1}{n-2}\int_x u_2(\log_\epsilon x)^{-n+2},\ \text{\&c.};$$

$$\therefore\ \int_x u\,(\log_\epsilon x)^{-n}=-\frac{ux}{n-1}(\log_\epsilon x)^{-n+1}-\frac{u_1x\,(\log_\epsilon x)^{-n+2}}{(n-1)(n-2)}$$

$$-\frac{u_2x\,(\log_\epsilon x)^{-n+3}}{(n-1)(n-2)(n-3)}-\text{\&c.}$$

$$+\frac{1}{(n-1)(n-2)\ldots 3.2.1}\int_x u_{n-1}\,(\log_\epsilon x)^{-1}.$$

Ex. 1. $\int_x x^m (\log_\epsilon x)^{-n}$;

in this case $ux = x^{m+1}$,

$\therefore\ u_1=(m+1)x^m,\ u_2=(m+1)^2x^m$, &c. $u_{n-1}=(m+1)^{n-1}x^m$;

$$\therefore\ \int_x x^m (\log_\epsilon x)^{-n} = -\,x^{m+1}\left\{\frac{(\log_\epsilon x)^{-n+1}}{n-1} + \frac{(m+1)(\log_\epsilon x)^{-n+2}}{(n-1)(n-2)} + \frac{(m+1)^2(\log_\epsilon x)^{-n+3}}{(n-1)(n-2)(n-3)} + \text{\&c.}\right\} + \frac{(m+1)^{n-1}}{(n-1)(n-2)\ldots 2.1}\int_x \frac{x^m}{\log_\epsilon x}.$$

Ex. 2. $\int_x \frac{x^m}{\log_\epsilon x}$;

this may be obtained in a series, by making $x^{m+1} = e^z$,

$$\therefore\ (m+1)\,x^m d_z x = e^z, \text{ and } (m+1)\log_\epsilon x = z;$$

$$\therefore\ \int_x \frac{x^m}{\log_\epsilon x} = \int_z \frac{e^z}{m+1}\cdot\frac{m+1}{z} = \int_z \frac{e^z}{z}$$

$$= \int_z \frac{1}{z}\left(1 + z + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \text{\&c.}\right)$$

$$= \log_\epsilon z + z + \frac{z^2}{1.2^2} + \frac{z^3}{1.2.3^2} + \text{\&c.}$$

$$= C + \log_\epsilon(\log_\epsilon x) + (m+1)\log_\epsilon x + \frac{\{(m+1)\log_\epsilon x\}^2}{1.2^2} + \text{\&c.},$$

including $\log_\epsilon(m+1)$ in the constant.

If $m = -1$, both the preceding integrals can be expressed in finite terms. For

$$\int_x \frac{1}{x(\log_\epsilon x)^n} = \int_x (\log_\epsilon x)^{-n} d_x(\log_\epsilon x) = -\frac{1}{n-1}\cdot\frac{1}{(\log_\epsilon x)^{n-1}}.$$

$$\int_x \frac{1}{x\log_\epsilon x} = \int_x \frac{d_x(\log_\epsilon x)}{\log_\epsilon x} = \log_\epsilon(\log_\epsilon x).$$

SECTION VI.

CIRCULAR FUNCTIONS.

71. We proceed next to the integration of differential coefficients involving circular functions of the variable.

$$\text{Since } d_x \sin u = \cos u\, d_x u,\ d_x \cos u = -\sin u\, d_x u,$$

$$d_x \tan u = (\sec u)^2 d_x u,\ d_x \cot u = -(\operatorname{cosec} u)^2 d_x u,$$

$$d_x \sec u = \sec u \tan u\, d_x u,\ d_x \operatorname{cosec} u = -\operatorname{cosec} u \cot u\, d_x u;$$

integrating, we have the fundamental formulæ,

$$\int_x \cos u\, d_x u = \sin u,$$

$$\int_x \sin u\, d_x u = -\cos u,$$

$$\int_x (\sec u)^2 d_x u = \int_x \frac{d_x u}{(\cos u)^2} = \tan u,$$

$$\int_x (\operatorname{cosec} u)^2 d_x u = \int_x \frac{d_x u}{(\sin u)^2} = -\cot u,$$

$$\int_x \sec u \tan u\, d_x u = \int_x \frac{\sin u\, d_x u}{(\cos u)^2} = \sec u,$$

$$\int_x \operatorname{cosec} u \cot u\, d_x u = \int_x \frac{\cos u\, d_x u}{(\sin u)^2} = -\operatorname{cosec} u.$$

72. Hence, changing u in the preceding formulæ into x, mx, $mx+a$, we find

$$\int_x \cos x = \sin x,\quad \int_x \sin x = -\cos x,\quad \int_x (\sec x)^2 = \tan x;$$

$$\int_x \cos mx = \frac{1}{m}\int_x \cos mx\,.\,d_x(mx) = \frac{1}{m}\sin mx;$$

$$\int_x \sin(mx+a) = \frac{1}{m}\int_x \sin(mx+a)\,.\,d_x(mx+a) = -\frac{1}{m}\cos(mx+a).$$

73. The following are cases of frequent occurrence, which are immediately reducible to logarithmic, or other known forms.

$$\int_x \frac{d_x u}{\sin u} = \int_x \frac{d_x u}{2\sin\frac{u}{2}\cos\frac{u}{2}} = \int_x \frac{d_x u}{2\tan\frac{u}{2}\left(\cos\frac{u}{2}\right)^2}$$

$$= \int_x \frac{\frac{1}{2}\left(\sec\frac{u}{2}\right)^2 d_x u}{\tan\frac{u}{2}} = \int_x \frac{d_x \tan\frac{u}{2}}{\tan\frac{u}{2}} = \log_\epsilon\left(\tan\frac{u}{2}\right).$$

$$\int_x \frac{d_x u}{\cos u} = \int_x \frac{d_x\left(\frac{\pi}{2}+u\right)}{\sin\left(\frac{\pi}{2}+u\right)} = \log_\epsilon\left\{\tan\left(\frac{\pi}{4}+\frac{u}{2}\right)\right\}.$$

$$\int_x \frac{1}{\sin x} = \log_\epsilon\left(\tan\frac{x}{2}\right).$$

$$\int_x \frac{1}{\cos x} = \log_\epsilon\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}.$$

$$\int_x \tan x = -\int_x \frac{d_x \cos x}{\cos x} = -\log_\epsilon(\cos x).$$

$$\int_x (\tan x)^2 = \int_x \left\{(\sec x)^2 - 1\right\} = \tan x - x.$$

$$\int_x \cot x = \int_x \frac{d_x \sin x}{\sin x} = \log_\epsilon(\sin x).$$

$$\int_x \frac{1}{\sin x \cos x} = \int_x \frac{(\sec x)^2}{\tan x} = \log_\epsilon(\tan x).$$

$$\int_x \frac{1}{(\sin x)^2(\cos x)^2} = \int_x \frac{(\sin x)^2 + (\cos x)^2}{(\sin x)^2(\cos x)^2}$$

$$= \int_x \left\{\frac{1}{(\cos x)^2} + \frac{1}{(\sin x)^2}\right\} = \tan x - \cot x.$$

$$\int_x \frac{1}{(\cos x)^4} = \int_x (\sec x)^2 . (\sec x)^2$$

$$= \int_x \left\{1 + (\tan x)^2\right\} . d_x \tan x = \tan x + \tfrac{1}{3} (\tan x)^3.$$

$$\int_x \frac{1}{\text{versin}\, x} = \int_x \frac{1}{2\left(\sin\frac{x}{2}\right)^2} = \int_x \left(\text{cosec}\,\frac{x}{2}\right)^2 d_x\left(\frac{x}{2}\right) = -\cot\frac{x}{2}.$$

74. The expression

$$(\cos x)^n (\sin x)^m$$

is immediately integrable, if either m or n be a positive odd integer, or if the sum of m and n be a negative even integer.

Let $n = 2r + 1$; make $\sin x = z$,

$$\therefore \int_x (\cos x)^{2r+1} (\sin x)^m = \int_x \left\{1 - (\sin x)^2\right\}^r . (\sin x)^m . d_x \sin x$$
$$= \int_z (1 - z^2)^r z^m.$$

Let $m = 2r + 1$; make $\cos x = z$,

$$\therefore \int_x (\cos x)^n (\sin x)^{2r+1} = -\int_x (\cos x)^n . \left\{1 - (\cos x)^2\right\}^r . d_x \cos x$$
$$= -\int_z z^n (1 - z^2)^r.$$

Let $m + n = -2r$; make $\tan x = z$,

$$\therefore \int_x (\cos x)^n (\sin x)^m = \int_x (\tan x)^m (\cos x)^{m+n} = \int_x (\tan x)^m (\sec x)^{2r}$$
$$= \int_x (\tan x)^m . (\sec x)^{2r-2} . d_x \tan x = \int_z z^m (1 + z^2)^{r-1}.$$

It is manifest that in each of these cases, upon expanding, the integration can be performed.

This, of course, includes the integrals

$$\int_x (\sin x)^{2m+1}, \quad \int_x (\cos x)^{2m+1}, \quad \int_x \frac{1}{(\sin x)^{2m}}, \quad \int_x \frac{1}{(\cos x)^{2m}}.$$

Ex. 1. $$\int_x (\sin x)^2 (\cos x)^5$$

$$= \int_x (\sin x)^2 \left\{1-(\sin x)^2\right\}^2 . d_x \sin x$$

$$= \int_x \left\{(\sin x)^2 - 2(\sin x)^4 + (\sin x)^6\right\} . d_x \sin x$$

$$= \frac{1}{3}(\sin x)^3 - \frac{2}{5}(\sin x)^5 + \frac{1}{7}(\sin x)^7.$$

Ex. 2. $$\int_x \frac{1}{(\sin x)^6 (\cos x)^4}$$

$$= \int_x \frac{(\sec x)^{10}}{(\tan x)^6} = \int_x \frac{(\sec x)^8 . d_x \tan x}{(\tan x)^6} = \int_z \frac{(1+z^2)^4}{z^6}, \text{ making } \tan x = z,$$

$$= \int_z (z^{-6} + 4z^{-4} + 6z^{-2} + 4 + z^2)$$

$$= -\frac{1}{5}(\tan x)^{-5} - \frac{4}{3}(\tan x)^{-3} - 6(\tan x)^{-1} + 4\tan x + \frac{1}{3}(\tan x)^3.$$

Ex. 3. $$\int_x (\tan x)^m = \int_z \frac{z^m}{z^2+1} \text{ (making } \tan x = z)$$

$$= \int_z (z^{m-2} - z^{m-4} + z^{m-6} - \&c.), \text{ by division;}$$

$$\therefore \int_x (\tan x)^m = \frac{(\tan x)^{m-1}}{m-1} - \frac{(\tan x)^{m-3}}{m-3} + \frac{(\tan x)^{m-5}}{m-5} - \&c.,$$

the last term being $(-1)^{\frac{m}{2}} . x$ (m even),

or $(-1)^{\frac{m-1}{2}} . \log_\epsilon (\sec x)$, ($m$ odd).

75. When none of the conditions of the last article are satisfied, the reduction of the integral

$$\int_x (\sin x)^m (\cos x)^n,$$

may be effected in the same manner as that of $\int_x x^{m-1}(a+bx^n)^p$ of which it is a particular case, for by making $\sin x = z$ it becomes $\int_z z^m (1-z^2)^{\frac{n-1}{2}}$. Now this last integral can be made to depend upon another, in which the index of z or $\sin x$ is altered by 2, and the index of $(1 - z^2)$ by 1, and therefore that of $\sqrt{1 - z^2}$ or $\cos x$ by 2. Hence the proposed integral can be made to depend upon another of the same form, in which one of the indices is altered by 2; and the quantity to be differentiated will be of the form $(\sin x)^r (\cos x)^q$, obtained by taking the one of lower dimensions of the two expressions whose integrals are to be connected, and increasing each index by unity. If we had begun by substituting z for $\cos x$, the reasoning would have been precisely the same.

Ex. 1. $\int_x (\sin x)^m$, m being an even integer.

We must make it depend upon $\int_x (\sin x)^{m-2}$, and therefore assume

$$P = \cos x\, (\sin x)^{m-1};$$

$$\begin{aligned} \therefore\ d_x P &= (\sin x)^{m-2} \left\{(m-1)\, (\cos x)^2 - (\sin x)^2\right\} \\ &= (\sin x)^{m-2} \left\{m - 1 - m\, (\sin x)^2\right\} \\ &= (m-1)\, (\sin x)^{m-2} - m\, (\sin x)^m; \end{aligned}$$

$$\therefore\ \int_x (\sin x)^m = -\frac{\cos x\, (\sin x)^{m-1}}{m} + \frac{m-1}{m} \int_x (\sin x)^{m-2}.$$

Hence to find $\int_x (\sin x)^4$, making $m = 4$, 2,

$$\int_x (\sin x)^4 = -\frac{\cos x\, (\sin x)^3}{4} + \tfrac{3}{4} \int_x (\sin x)^2,$$

$$\int_x (\sin x)^2 = -\frac{\cos x \sin x}{2} + \tfrac{1}{2}\, x;$$

$$\therefore\ \int_x (\sin x)^4 = -\frac{\cos x\, (\sin x)^3}{4} - \frac{3 \cos x \sin x}{8} + \frac{3x}{8}.$$

Ex. 2. $\int_x (\sin x)^m (\cos x)^n$, m and n being even integers.

To make this depend upon $\int_x (\sin x)^m (\cos x)^{n-2}$,

$$\text{Assume } P = (\sin x)^{m+1} (\cos x)^{n-1};$$

$$\therefore\ d_x P = (\sin x)^m (\cos x)^{n-2} \{(m+1)(\cos x)^2 - (n-1)(\sin x)^2\}$$

$$= (\sin x)^m (\cos x)^{n-2} \{(m+n)(\cos x)^2 - (n-1)\}$$

$$= (m+n)(\sin x)^m (\cos x)^n - (n-1)(\sin x)^m (\cos x)^{n-2};$$

$$\therefore\ \int_x (\sin x)^m (\cos x)^n$$

$$= \frac{(\sin x)^{m+1} (\cos x)^{n-1}}{m+n} + \frac{n-1}{m+n} \int_x (\sin x)^m (\cos x)^{n-2}.$$

Hence changing n into $n-2$, $n-4$, &c.

$$\int_x (\sin x)^m (\cos x)^{n-2}$$

$$= \frac{(\sin x)^{m+1} (\cos x)^{n-3}}{m+n-2} + \frac{n-3}{m+n-2} \int_x (\sin x)^m (\cos x)^{n-4}, \text{ \&c.};$$

till at last, since n is even, we come to

$$\int_x (\sin x)^m (\cos x)^2 = \frac{(\sin x)^{m+1} \cos x}{m+2} + \frac{1}{m+2} \int_x (\sin x)^m;$$

therefore, collecting the results,

$$\int_x (\sin x)^m (\cos x)^n = \frac{(\sin x)^{m+1}}{m+n} \Big\{(\cos x)^{n-1} + \frac{(n-1)(\cos x)^{n-3}}{m+n-2}$$

$$+ \frac{(n-1)(n-3)}{(m+n-2)(m+n-4)} (\cos x)^{n-5} + \text{\&c.}\Big\}$$

$$+ \frac{(n-1)(n-3)\ldots 3.1}{(m+n)(m+n-2)\ldots(m+2)} \cdot \int_x (\sin x)^m;$$

and $\int_x (\sin x)^m$ is known from Ex. 1.

Ex. 3. $\int_x \frac{1}{(\cos x)^n}$, n being an odd integer.

To make this depend upon $\int_x (\cos x)^{-n+2}$,

assume $P = \sin x\,(\cos x)^{-n+1}$, and we find

$$\int_x \frac{1}{(\cos x)^n} = \sin x \left\{\frac{(\cos x)^{-n+1}}{n-1} + \frac{(n-2)\,(\cos x)^{-n+3}}{(n-1)\,(n-3)} + \&c.\right\}$$

$$+ \frac{(n-2)\,(n-4)\ldots 3\,.\,1}{(n-1)\,(n-3)\ldots 4\,.\,2}\,.\,\log_e \left\{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right\}.$$

76. We can also make $\int_x (\sin x)^m\,(\cos x)^n$ depend upon an integral of the same form in which both the indices are altered. For, integrating by parts,

$$\int_x (\sin x)^m\,(\cos x)^n = \int_x (\sin x)^m\,(\cos x)^{n-1}\,d_x \sin x$$

$$= \frac{(\sin x)^{m+1}\,(\cos x)^{n-1}}{m+1} + \frac{n-1}{m+1}\int_x (\sin x)^{m+2}\,(\cos x)^{n-2}.$$

Ex. $\int_x \frac{(\cos x)^6}{(\sin x)^4}$.

$$= -\frac{1}{3}\,\frac{(\cos x)^5}{(\sin x)^3} + \frac{5}{3}\,\frac{(\cos x)^3}{\sin x} + \frac{5}{2}\cos x \sin x + \frac{5}{2}\,x.$$

O **77.** The integrals $\int_x (\cos x)^n$, $\int_x (\sin x)^n$, may also be obtained by substituting for the powers of $\cos x$ and $\sin x$, their values in terms of the simple dimensions of sines or cosines of multiple angles. Thus,

$$2^{n-1}\,(\cos x)^n = \cos nx + n\cos(n-2)\,x + \frac{n\,(n-1)}{1\,.\,2}\cos(n-4)\,x + \&c.$$

$$\therefore \int_x (\cos x)^n = \frac{1}{2^{n-1}} \left\{ \frac{\sin n x}{n} + \frac{n \sin (n-2) x}{n-2} \right.$$

$$\left. + \frac{n(n-1)}{1.2} \; \frac{\sin (n-4) x}{n-4} + \&\text{c.} \right\};$$

the last term being $\dfrac{1}{2^{\frac{n}{2}}} \; \dfrac{1.3.5\ldots(n-1)}{1.2.3\ldots\frac{n}{2}} \, x$, ($n$ even),

and $\dfrac{1}{2^{n-1}} \; \dfrac{n(n-1)\ldots\frac{n+3}{2}}{1.2.3\ldots\frac{n-1}{2}} \sin x$, ($n$ odd).

Similarly, may $\int_x (\sin x)^n$ be found from the series

$$(-1)^{\frac{n}{2}} 2^{n-1} (\sin x)^n = \cos n x - n \cos (n-2) x + \frac{n(n-1)}{1.2} \cos (n-4) x$$

$$+ \&\text{c.} + \frac{(-1)^{\frac{n}{2}}}{2} \; \frac{n(n-1)\ldots\left(\frac{n}{2}+1\right)}{1.2.3\ldots\frac{n}{2}}, \ (n \text{ even});$$

and $$(-1)^{\frac{n-1}{2}} 2^{n-1} (\sin x)^n = \sin n x - n \sin (n-2) x + \frac{n(n-1)}{1.2} \sin (n-4) x$$

$$+ \&\text{c.} + (-1)^{\frac{n-1}{2}} . \frac{n(n-1)\ldots\frac{n+3}{2}}{1.2.3\ldots\frac{n-1}{2}} \sin x, \ (n \text{ odd}).$$

Ex. 1. $\int_x (\cos x)^4 = \dfrac{1}{32} \sin 4x + \dfrac{1}{4} \sin 2x + \dfrac{3}{8} x.$

Ex. 2. $\int_x (\sin x)^4 = \dfrac{1}{32} \sin 4x - \dfrac{1}{4} \sin 2x + \dfrac{3}{8} x.$

78. To integrate

$$\sin mx \cos nx, \quad \sin mx \sin nx, \quad \cos mx \cos nx.$$

Since $\sin mx \cos nx = \frac{1}{2}\{\sin(m+n)x + \sin(m-n)x\}$,

$$\therefore \int_x \sin mx \cos nx = -\frac{1}{2}\left\{\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right\} + C.$$

Also, $\sin mx \sin nx = \frac{1}{2}\{\cos(m-n)x - \cos(m+n)x\}$,

$$\therefore \int_x \sin mx \sin nx = \frac{1}{2}\left\{\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right\} + C.$$

Similarly, $\int_x \cos mx \cos nx = \frac{1}{2}\left\{\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right\}$.

In the same manner, may the product of any number of the sines and cosines of multiples of an angle be integrated.

79. Expressions like the above may also be integrated by parts; and the result so obtained appears under a somewhat simpler form. Thus, in the general formula, make

$$U = \cos(nx+\beta), \quad d_x V = \cos(mx+a);$$

$$\therefore \int_x \cos(mx+a)\cos(nx+\beta) = \frac{1}{m}\sin(mx+a)\cos(nx+\beta)$$

$$+\frac{n}{m}\int_x \sin(mx+a)\sin(nx+\beta);$$

similarly, $\int_x \sin(mx+a)\sin(nx+\beta) = -\frac{1}{m}\cos(mx+a)\sin(nx+\beta)$

$$+\frac{n}{m}\int_x \cos(mx+a)\cos(nx+\beta);$$

Hence, substituting and transposing,

$$\left(1-\frac{n^2}{m^2}\right)\int_x \cos(mx+a)\cos(nx+\beta)$$

$$=\frac{1}{m}\sin(mx+a)\cos(nx+\beta)-\frac{n}{m^2}\cos(mx+a)\sin(nx+\beta),$$

$$\text{or } \int_x \cos(mx+a)\cos(nx+\beta)$$

$$=\frac{m\sin(mx+a)\cos(nx+\beta)-n\cos(mx+a)\sin(nx+\beta)}{m^2-n^2}.$$

80. To integrate $\dfrac{1}{a+b\cos x}$.

$$\int_x \frac{1}{a+b\cos x}=\int_x \frac{1}{a\left\{\left(\cos\frac{x}{2}\right)^2+\left(\sin\frac{x}{2}\right)^2\right\}+b\left\{\left(\cos\frac{x}{2}\right)^2-\left(\sin\frac{x}{2}\right)^2\right\}}$$

$$=\int_x \frac{\left(\sec\frac{x}{2}\right)^2}{a+b+(a-b)\left(\tan\frac{x}{2}\right)^2}=2\int_x \frac{d_x\left(\tan\frac{x}{2}\right)}{a+b+(a-b)\left(\tan\frac{x}{2}\right)^2}$$

$$=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\tan\frac{x}{2}\sqrt{\frac{a-b}{a+b}}\right), \text{ if } a>b.$$

But if $a<b$, $\displaystyle\int_x \frac{1}{a+b\cos x}=2\int_x \frac{d_x\left(\tan\frac{x}{2}\right)}{b+a-(b-a)\left(\tan\frac{x}{2}\right)^2}$

$$=\frac{1}{\sqrt{b^2-a^2}}\log_\epsilon\left(\frac{\sqrt{b+a}+\tan\frac{x}{2}\sqrt{b-a}}{\sqrt{b+a}-\tan\frac{x}{2}\sqrt{b-a}}\right).$$

81. To integrate $\dfrac{1}{a+b\sin x}$.

$$\int_x \frac{1}{a+b\sin x} = \int_x \frac{1}{a\left\{\left(\cos\frac{x}{2}\right)^2 + \left(\sin\frac{x}{2}\right)^2\right\} + 2b\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \int_x \frac{a\left(\sec\frac{x}{2}\right)^2}{a^2 + a^2\left(\tan\frac{x}{2}\right)^2 + 2ab\tan\frac{x}{2}} = 2\int_x \frac{d_x\left(a\tan\frac{x}{2}+b\right)}{a^2 - b^2 + \left(a\tan\frac{x}{2}+b\right)^2}$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}\right), \text{ if } a > b.$$

But if $a < b$, $\displaystyle\int_x \frac{1}{a+b\sin x} = 2\int_x \frac{d_x\left(a\tan\frac{x}{2}+b\right)}{\left(a\tan\frac{x}{2}+b\right)^2 - (b^2-a^2)}$

$$= \frac{1}{\sqrt{b^2-a^2}}\log_e\left(\frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}\right).$$

Also $\displaystyle\int_x \frac{1}{a+b\cos x + c\sin x} = \int_x \frac{1}{a+e\cos(x-\alpha)}$, $\left(\text{if } e = \sqrt{b^2+c^2} \text{ and } \tan\alpha = \frac{c}{b}\right)$, and therefore falls under the preceding form.

20

82. To integrate $\dfrac{1}{(a+b\cos x)^n}$.

If we make $a + b\cos x = z$, the proposed integral is transformed into $-\displaystyle\int_z \frac{1}{z^n\sqrt{b^2-a^2+2az-z^2}}$, a formula of reduction for which can be obtained (Art. 58.) by differentiating

$$z^{-n+1}\sqrt{b^2-(z-a)^2} \text{ or } \sin x(a+b\cos x)^{-n+1}.$$

Hence, replacing x by X,

$$d_x(\sin x X^{-n+1}) = X^{-n}\{X\cos x + (n-1)b(\sin x)^2\}$$

$$= \frac{X^{-n}}{b}\{Xb\cos x + (n-1)(b^2 - (b\cos x)^2)\};$$

and eliminating $b\cos x$ from the quantity within brackets by the equation $b\cos x = X - a$, we find

$$d_x(\sin x X^{-n+1}) = \frac{X^{-n}}{b}\{X(X-a) + (n-1)b^2 - (n-1)(X-a)^2\}$$

$$= \frac{X^{-n}}{b}\{-(n-2)X^2 + (2n-3)aX - (n-1)(a^2-b^2)\};$$

therefore, integrating and transposing,

$$\int_x \frac{1}{X^n} = \frac{-b\sin x}{(n-1)(a^2-b^2)X^{n-1}}$$

$$+ \frac{(2n-3)a}{(n-1)(a^2-b^2)}\int_x \frac{1}{X^{n-1}} - \frac{n-2}{(n-1)(a^2-b^2)}\int_x \frac{1}{X^{n-2}}.$$

When n is a positive integer, by means of this formula of reduction, the integral may be made to depend upon the known form $\int_x \frac{1}{a + b\cos x}$.

Ex. $\int_x \frac{1}{(a + b\cos x)^2}$.

Here the quantity to be differentiated is $\frac{\sin x}{X}$; and we find

$$\int_x \frac{1}{X^2} = \frac{1}{a^2-b^2}\left\{\frac{-b\sin x}{a+b\cos x} + \frac{2a}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)\right\}.$$

83. Hence also we can find

$$\int_x \frac{a_1 + b_1\cos x}{(a+b\cos x)^n}, \text{ and } \int_x \frac{a_1 + b_1\sin x}{(a+b\cos x)^n}.$$

$$\text{For } \int_x \frac{a_1 + b_1 \cos x}{(a + b\cos x)^n} = \int_x \frac{a_1 - \dfrac{b_1 a}{b} + \dfrac{b_1}{b}(a + b\cos x)}{(a + b\cos x)^n}$$

$$= \frac{ba_1 - ab_1}{b} \int_x \frac{1}{(a + b\cos x)^n} + \frac{b_1}{b} \int_x \frac{1}{(a + b\cos x)^{n-1}};$$

$$\text{and } \int_x \frac{a_1 + b_1 \sin x}{(a + b\cos x)^n} = \int_x \frac{b_1 \sin x}{(a + b\cos x)^n} + a_1 \int_x \frac{1}{(a + b\cos x)^n}$$

$$= \frac{b_1}{(n-1)\,b}\,\frac{1}{(a + b\cos x)^{n-1}} + a_1 \int_x \frac{1}{(a + b\cos x)^n}.$$

84. To integrate $e^{ax}\sin mx$, $e^{ax}\cos mx$.

These are easily integrated by parts, thus,

$$\int_x e^{ax}\sin mx = \frac{1}{a}\,e^{ax}\sin mx - \frac{m}{a}\int_x e^{ax}\cos mx$$

$$= \frac{1}{a}\,e^{ax}\sin mx - \frac{m}{a}\left(\frac{1}{a}\,e^{ax}\cos mx + \frac{m}{a}\int_x e^{ax}\sin mx\right);$$

$$\therefore \left(1 + \frac{m^2}{a^2}\right)\int_x e^{ax}\sin mx = e^{ax}\,\frac{a\sin mx - m\cos mx}{a^2},$$

$$\text{or } \int_x e^{ax}\sin mx = e^{ax}\,\frac{a\sin mx - m\cos mx}{m^2 + a^2}.$$

$$\text{Similarly, } \int_x e^{ax}\cos mx = e^{ax}\,\frac{a\cos mx + m\sin mx}{m^2 + a^2}.$$

Hence also we can integrate

$$e^{ax}\sin mx\cos nx = e^{ax}.\tfrac{1}{2}\left\{\sin(m+n)\,x + \sin(m-n)\,x\right\}.$$

85. To integrate $e^{ax}(\sin x)^n$, $e^{ax}(\cos x)^n$.

The integrals of these may be made to depend upon

$$\int_x e^{ax}(\sin x)^{n-2}, \quad \text{and} \quad \int_x e^{ax}(\cos x)^{n-2},$$

by integration by parts; thus,

$$\int_x e^{ax}(\sin x)^n = \frac{1}{a}e^{ax}(\sin x)^n - \frac{n}{a}\int_x e^{ax}(\sin x)^{n-1}\cos x;$$

$$\text{but } \int_x e^{ax}(\sin x)^{n-1}\cos x = \frac{e^{ax}}{a}(\sin x)^{n-1}\cos x$$

$$-\frac{1}{a}\int_x e^{ax}\left\{(n-1)(\sin x)^{n-2}(\cos x)^2 - (\sin x)^n\right\},$$

$$= \frac{e^{ax}}{a}(\sin x)^{n-1}\cos x - \frac{1}{a}\int_x e^{ax}\left\{(n-1)(\sin x)^{n-2} - n(\sin x)^n\right\};$$

$$\therefore \int_x e^{ax}(\sin x)^n = \frac{1}{a}e^{ax}(\sin x)^n - \frac{n}{a^2}e^{ax}(\sin x)^{n-1}\cos x$$

$$+\frac{n(n-1)}{a^2}\int_x e^{ax}(\sin x)^{n-2} - \frac{n^2}{a^2}\int_x e^{ax}(\sin x)^n,$$

$$\text{or } \int_x e^{ax}(\sin x)^n = e^{ax}(\sin x)^{n-1}\frac{(a\sin x - n\cos x)}{n^2+a^2}$$

$$+\frac{n(n-1)}{n^2+a^2}\int_x e^{ax}(\sin x)^{n-2}.$$

$$\text{Similarly, } \int_x e^{ax}(\cos x)^n = e^{ax}(\cos x)^{n-1}\frac{(a\cos x + n\sin x)}{n^2+a^2}$$

$$+\frac{n(n-1)}{n^2+a^2}\int_x e^{ax}(\cos x)^{n-2}.$$

86. The two integrals of the preceding article may also be obtained, by substituting for $(\cos x)^n$ and $(\sin x)^n$, their developments in sines and cosines of multiple angles, according to the formulæ of Art. 77.

Hence the general term of $\int_x e^{ax}(\cos x)^n$, is

$$\frac{1}{2^{n-1}}\frac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots\ldots r}\int_x e^{ax}\cos(n-2r)x,$$

$$\text{which } = \frac{1}{2^{n-1}}\frac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots\ldots r}\times$$

$$e^{ax}\frac{a\cos(n-2r)x+(n-2r)\sin(n-2r)x}{a^2+(n-2r)^2};$$

and to obtain all the terms, (r) must be taken from 0 to $\frac{n}{2}$, when n is even, and to $\frac{n-1}{2}$, when n is odd.

87. It may be observed that all functions of $\sin x$ and $\cos x$, may be converted into exponential functions, by putting for those quantities their exponential values, viz.

$$\frac{1}{2\sqrt{-1}}\left(e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}\right), \text{ and } \tfrac{1}{2}\left(e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}\right).$$

By this transformation, some expressions may be readily integrated.

$$\text{Ex. } \int_x e^{ax}\sin mx = \frac{1}{2\sqrt{-1}}\int_x\left(e^{(a+m\sqrt{-1})x}-e^{(a-m\sqrt{-1})x}\right)$$

$$=\frac{1}{2\sqrt{-1}}\left(\frac{e^{(a+m\sqrt{-1})x}}{a+m\sqrt{-1}}-\frac{e^{(a-m\sqrt{-1})x}}{a-m\sqrt{-1}}\right)$$

$$=\frac{e^{ax}}{a^2+m^2}\left(a.\frac{e^{mx\sqrt{-1}}-e^{-mx\sqrt{-1}}}{2\sqrt{-1}}-m.\frac{e^{mx\sqrt{-1}}+e^{-mx\sqrt{-1}}}{2}\right)$$

$$=\frac{e^{ax}(a\sin mx-m\cos mx)}{a^2+m^2}.$$

88. To obtain the integral of $e^{ax}(\sin x)^n(\cos x)^m$, we must endeavour to express $(\sin x)^n(\cos x)^m$, by simple dimensions of cosines or sines of multiples of x.

$$\text{Ex.}\quad \int_x e^x(\sin x)^5(\cos x)^2;$$

$$\text{let } 2\cos x = z + z^{-1},$$

$$\therefore\ (2\sqrt{-1}\sin x)^5(2\cos x)^2 = (z - z^{-1})^5(z + z^{-1})^2$$

$$= (z - z^{-1})^3(z^2 - z^{-2})^2 = (z^7 - z^{-7}) - 3(z^5 - z^{-5})$$

$$+ (z^3 - z^{-3}) + 5(z - z^{-1}),$$

$$\therefore\ 2^6(\sin x)^5(\cos x)^2 = \sin 7x - 3\sin 5x + \sin 3x + 5\sin x;$$

and the proposed expression is transformed into

$$\frac{e^x}{2^6}(\sin 7x - 3\sin 5x + \sin 3x + 5\sin x),$$

each term of which is integrable by Art. 84.

89. All such expressions as $\int_x u\sin^{-1}v$, $\int_x u\tan^{-1}v$, &c. where u and v represent functions of x, must be integrated by parts.

$$\text{Ex. 1.}\quad \int_x u\sin^{-1}v = \sin^{-1}v\int_x u - \int_x \frac{\int_x u \,.\, d_x v}{\sqrt{1 - v^2}}.$$

$$\text{Hence}\ \int_x \frac{\sin^{-1}x}{(1 - x^2)^{\frac{3}{2}}} = \frac{x\sin^{-1}x}{\sqrt{1 - x^2}} + \log_e\sqrt{1 - x^2};$$

$$\text{and}\ \int_x x^n\sin^{-1}x = \frac{x^{n+1}}{n+1}\sin^{-1}x - \frac{1}{n+1}\int_x \frac{x^{n+1}}{\sqrt{1 - x^2}}.$$

$$\text{Ex. 2.}\quad \int_x u\tan^{-1}v = \tan^{-1}v\int_x u - \int_x \frac{\int_x u \,.\, d_x v}{1 + v^2}.$$

$\therefore \int_x \tan^{-1}x \cdot \frac{d}{dx}\tan^{-1}x = \frac{(\tan^{-1}x)^2}{2}$

Hence $$\int_x \frac{x^2 \tan^{-1} x}{1+x^2} = \tan^{-1} x \int_x \frac{x^2}{1+x^2} - \int_x \left(\frac{1}{1+x^2} \int \frac{x^2}{1+x^2}\right)$$

$$= \tan^{-1} x\,(x - \tan^{-1} x) - \int_x \frac{1}{1+x^2}\,(x - \tan^{-1} x)$$

[illegible]

$\sqrt{1+x^2}$

$+x^2$

$$= \tan^{-1} x\,(x - \tan^{-1} x) - \log_e \sqrt{1+x^2} + \tfrac{1}{2}(\tan^{-1} x)^2$$

$$= \tan^{-1} x\,(x - \tfrac{1}{2}\tan^{-1} x) - \log_e \sqrt{1+x^2}.$$

Also $$\int_x \tan^{-1} \sqrt{\frac{x}{a}} = (x+a)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{a x}.$$

90. In like manner the expressions $x^n \cos x$, $x^n \sin x$, must be integrated by parts.

Ex. 1. $\int_x x^2 \cos x = x^2 \sin x - 2 \int_x x \sin x$

$$= x^2 \sin x - 2\,(-x\cos x + \sin x) = x^2 \sin x + 2x\cos x - 2\sin x.$$

And generally

$$\int_x x^n \cos x = \sin x \,\{x^n - n(n-1)\,x^{n-2} + n\ldots(n-3)\,x^{n-4} - \&c.\}$$
$$+ \cos x\,\{n x^{n-1} - n(n-1)(n-2)\,x^{n-3} + n\ldots(n-4)\,x^{n-5} - \&c.\}.$$

Ex. 2. $\int_x x^3 \sin x = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x.$

And generally

$$\int_x x^n \sin x = -\cos x\,\{x^n - n(n-1)\,x^{n-2} + n\ldots(n-3)\,x^{n-4} - \&c.\}$$
$$+ \sin x\,\{n x^{n-1} - n(n-1)(n-2)\,x^{n-3} + n\ldots(n-4)\,x^{n-5} - \&c.\},$$

the last terms of the series within the brackets being

$(-1)^{\frac{n}{2}} n(n-1)\ldots3.2.1$, and $-(-1)^{\frac{n}{2}} n(n-1)\ldots3.2x$, n even;

and $(-1)^{\frac{n-1}{2}} n(n-1)\ldots 3.2x$, and $(-1)^{\frac{n-1}{2}} n(n-1)\ldots 3.2.1$, n odd.

If n be negative, we have

$$\int_x x^{-n} \sin x = -\frac{x^{-n+1} \sin x}{n-1} - \frac{x^{-n+2} \cos x}{(n-1)(n-2)}$$

$$+\frac{x^{-n+3} \sin x}{(n-1)(n-2)(n-3)} + \&\text{c.}$$

the last term involving $\int_x \frac{\sin x}{x}$, or $\int_x \frac{\cos x}{x}$, which can be integrated only in infinite series by expanding $\sin x$ and $\cos x$.

91. The following integrals may be reduced to those in the preceding article.

1. $\int_x (\sin^{-1} x)^n$, by making $\sin^{-1} x = z$; for it becomes

$$\int_z (\sin^{-1} x)^n d_z x = \int_z z^n \cos z.$$

2. $\int_x x^n (\cos x)^m$, and $\int_x x^n (\sin x)^m$, by substituting for $(\cos x)^m$ and $(\sin x)^m$ their developments in simple dimensions of the sines and cosines of multiples of x; for then each term will be of the form, $ax^n \cos rx$, or $ax^n \sin rx$.

Ex. $\int_x x^2 (\cos x)^3 = \frac{1}{4} \int_x x^2 (\cos 3x + 3 \cos x)$

$$= \frac{1}{108} \int_x (3x)^2 \cos 3x d_x (3x) + \tfrac{3}{4} \int_x x^2 \cos x$$

$$= \frac{1}{108} \left\{(3x)^2 \sin 3x + 2 \,.\, 3x \,.\, \cos 3x - 2 \sin 3x\right\}$$

$$+ \tfrac{3}{4} (x^2 \sin x + 2x \cos x - 2 \sin x).$$

3. $\int_x \frac{x^n \sin^{-1} x}{\sqrt{1-x^2}}$, by making $\sin^{-1} x = z$; for it then becomes $\int_x x^n \sin^{-1} x d_x (\sin^{-1} x) = \int_z (\sin z)^n . z$, and so is reduced to the preceding case.

92. The following integrals, involving circular functions, deserve notice, as several of them occur in the application of Mathematics to Natural Philosophy.

1. $$\int_x \frac{1}{a(\cos x)^2 + b(\sin x)^2}$$

$$= \int_x \frac{d_x \tan x}{a + b(\tan x)^2} = \frac{1}{\sqrt{ab}} \tan^{-1}\left(\tan x \sqrt{\frac{b}{a}}\right).$$

2. $$\int_x \frac{1}{a + b\tan x} = \frac{1}{a^2+b^2} \int_x \left\{b . \frac{-a\sin x + b\cos x}{a\cos x + b\sin x} + a\right\}$$

$$= \frac{1}{a^2+b^2}\left\{b \log_\epsilon (a\cos x + b\sin x) + ax\right\}.$$

3. $$\int_x \frac{1}{a + b(\sin x)^2}$$

$$= \int_x \frac{-d_x \cot x}{a + b + a(\cot x)^2} = \frac{1}{\sqrt{a^2+ab}} \cot^{-1}\left(\frac{\cot x \sqrt{a}}{\sqrt{a+b}}\right).$$

4. $$\int_x \frac{\sin x (\cos x)^2}{1 + c^2(\cos x)^2} = \frac{1}{c^2} \int_x \frac{\sin x \{1 + c^2(\cos x)^2 - 1\}}{1 + c^2(\cos x)^2}$$

$$= \frac{1}{c^2} \int_x \left(\sin x + \frac{d_x \cos x}{1 + c^2(\cos x)^2}\right) = -\frac{\cos x}{c^2} + \frac{1}{c^3} \tan^{-1}(c\cos x).$$

5. $$\int_x \frac{(\cos x)^3}{1 - c^2(\cos x)^2} = \frac{1}{c^2}\int_x \frac{\cos x\,\{1 - 1 + c^2(\cos x)^2\}}{1 - c^2(\cos x)^2}$$

$$= \frac{1}{c^2}\int_x \left\{\frac{d_x \sin x}{1 - c^2 + c^2(\sin x)^2} - \cos x\right\}$$

$$= \frac{1}{c^3\sqrt{1 - c^2}}\tan^{-1}\left(\frac{c\sin x}{\sqrt{1 - c^2}}\right) - \frac{\sin x}{c^2}.$$

6. $$\int_x \frac{\tan x}{\sqrt{a + b(\tan x)^2}} = \int_x \frac{\sin x}{\sqrt{a(\cos x)^2 + b(\sin x)^2}}$$

$$= \frac{1}{\sqrt{b - a}}\int_x \frac{-d_x(\sqrt{b - a}\,.\cos x)}{\sqrt{b - (b - a)(\cos x)^2}}$$

$$= \frac{1}{\sqrt{b - a}}\cos^{-1}\left(\frac{\sqrt{b - a}\cos x}{\sqrt{b}}\right).$$

7. $$\int_x \frac{1}{\sin x}\sqrt{a + b(\sin x)^2} = \int_x \frac{a + b(\sin x)^2}{\sin x\sqrt{a + b(\sin x)^2}}$$

$$= \int_x \left\{\frac{a}{(\sin x)^2\sqrt{a(\operatorname{cosec} x)^2 + b}} + \frac{b\sin x}{\sqrt{a + b(\sin x)^2}}\right\}$$

$$= \int_x \left\{-\frac{a d_x \cot x}{\sqrt{a + b + a(\cot x)^2}} - \frac{b d_x \cos x}{\sqrt{a + b - b(\cos x)^2}}\right\}$$

$$= -\sqrt{a}\log_\epsilon\left\{\sqrt{a}\cot x + \sqrt{a(\operatorname{cosec} x)^2 + b}\right\}$$

$$+ \sqrt{b}\cos^{-1}\left(\frac{\sqrt{b}\cos x}{\sqrt{a + b}}\right).$$

93. There are several algebraic expressions, which are conveniently integrated by the introduction of circular functions.

1. $$\int_x \frac{1}{(b^2+x^2)\sqrt{a^2+x^2}}$$

$$= \frac{1}{b}\int_x \frac{d_x \tan^{-1}\frac{x}{b}}{\sqrt{a^2+x^2}} = \frac{1}{b}\int_z \frac{1}{\sqrt{a^2+b^2(\tan z)^2}} \ \left(\text{if } \frac{x}{b} = \tan z\right)$$

$$= \frac{1}{b}\int_z \frac{d_z \sin z}{\sqrt{a^2-(a^2-b^2)(\sin z)^2}} = \frac{1}{b\sqrt{a^2-b^2}}\sin^{-1}\left(\frac{\sin z\sqrt{a^2-b^2}}{a}\right)$$

$$= \frac{1}{b\sqrt{a^2-b^2}}\sin^{-1}\left(\frac{x\sqrt{a^2-b^2}}{a\sqrt{x^2+b^2}}\right).$$

2. $$\int_x \frac{\sqrt{1-x^2}}{(1+x)^2} = \int_x \frac{1}{\sqrt{1-x^2}}\cdot\frac{1-x}{1+x} = -\int_z\left(\tan\frac{z}{2}\right)^2 (\text{if } x=\cos z)$$

$$= \int_z\left\{1-\left(\sec\frac{z}{2}\right)^2\right\} = z - 2\tan\frac{z}{2} = \cos^{-1}x - 2\sqrt{\frac{1-x}{1+x}}.$$

SECTION VII.

INTEGRATION BETWEEN LIMITS, AND BY INFINITE SERIES.

Art. 94. When a proposed function cannot be integrated by any of the preceding methods, it must be developed in an infinite series, and each term separately integrated. Integration by series is of great importance, because the integrals which arise in the application of mathematics to the different branches of Natural Philosophy can frequently be obtained only by this process. Moreover they are usually required not in the state in which we have hitherto obtained them, where the variable and constant remain undetermined, but between limits: that is, the value of $\int_x u$ is required when $x = b$, under the condition that its value, corresponding to $x = a$, shall be 0, or a given quantity; in other words it is the *difference* of the values assumed by the integral, when for the variable two particular values are successively substituted, that is generally wanted; in taking this difference the arbitrary constant disappears, and a result is obtained in which no part is undetermined.

95. Hence, it will first be necessary to explain the method of correcting integrals, and of finding the values of *definite* integrals or integrals taken between given limits.

It has already been stated that an arbitrary constant must be added to every integral to make it complete; and it is often convenient to give the constant the same form, as the expression to which it is annexed, by which means the result is simplified; thus

$$\int_x \frac{n}{x} = n \log_\epsilon x - n \log_\epsilon c = n \log_\epsilon \frac{x}{c} = \log_\epsilon \left(\frac{x}{c}\right)^n.$$

$$\int_x \frac{1}{1 + x^2} = \tan^{-1} x + \tan^{-1} c = \tan^{-1} \frac{x + c}{1 - xc}.$$

It was also shewn how the value of the annexed constant might be determined, when corresponding values of the integral, and of the variable x, were known. This may be made clearer by the following illustration.

If A represent the area of a curve, contained by the arc, the ordinates at its extremities, and the intercepted portion of the axis of x, it is proved in the Differential Calculus that $d_x A = y$,

$$\therefore\ A = \int_x y.$$

Now let $y = f(x)$ be the equation to the curve BQ, Fig. 1. where $AM = x$, $MQ = y$,

$$\therefore\ \text{the area} = \int_x f(x) = \phi(x) + C, \text{ suppose.}$$

But if the area $PNMQ$ be required, commencing with the fixed ordinate PN for which $AN = a$, then when $x = AN = a$, QM coincides with PN, and the area vanishes,

$$\therefore\ 0 = \phi(a) + C, \quad \text{or } C = -\phi(a);$$

$$\therefore\ \text{area } PNMQ = \phi(x) - \phi(a);$$

$$\text{and if } AS = b, \quad \text{area } PNSR = \phi(b) - \phi(a).$$

The expression for the area $PNMQ$ is a *corrected* integral, that for the area $PNSR$ a *definite* integral, and the expression $\phi x + C$ an *indefinite* or general integral.

96. The quantity we have expressed by $\phi(a)$ may otherwise be expressed by $\int_{x=a} fx$, and this is the notation we shall employ.

Hence $\int_{x=a} u$ denotes what the integral of u with respect to x becomes, when in it a is substituted for x; and

$$\int_{x=b} u - \int_{x=a} u, \quad \text{or } \left(\int_{x=b} - \int_{x=a}\right) u$$

(as we shall in future write it) denotes the value $f(a, b)$ assumed by $\int_x u$, when in it b is substituted for x, the condition that it vanishes when $x = a$, having been previously introduced in determining the constant; it is called the definite integral of u between the limits $x = a$, $x = b$; and a and b are called respectively the inferior and superior limits. Since every function of x may be supposed to represent the ordinate of a curve, the problem of finding $\int_x u$ between the limits $x = a$, $x = b$, amounts to nothing more, than to find the area of such a curve included between the ordinates corresponding to $x = a$, $x = b$.

The sign of definite integration employed by Fourier and other writers is $\int_a^b{}_x$, the superior limit being placed uppermost.

97. On account of its great importance, we shall illustrate this matter still more particularly, by supposing the curve in Art. 95 to be a circle, and the origin of the co-ordinates in its center;

$$CN = x, \quad PN = y, \quad CB = a, \text{ Fig. } 2; \quad \therefore\ y = \sqrt{a^2 - x^2};$$

$$\therefore\ \text{area} = \int_x \sqrt{a^2 - x^2} = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$$

Now if the area of the portion $BCNP$ be required, that is, if the integral vanishes when $x = 0$, since the quantity

$$\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

also vanishes in that case, the above equation becomes

$$0 = 0 + C, \quad \therefore\ C = 0;$$

$$\therefore\ \text{area } BCNP = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

Hence, making $x = a$, the area of the quadrant BCA

$$= \frac{a^2}{2}\sin^{-1} 1 = \frac{a^2}{2}\,\frac{\pi}{2} = \frac{\pi a^2}{4};$$

that is, the definite integral of $\sqrt{a^2 - x^2}$ between the limits

$$x = 0, \quad x = a, \quad \text{or} \quad \left(\int_{x=a} - \int_{x=0}\right) \sqrt{a^2 - x^2} = \frac{\pi a^2}{4}.$$

Hence we see the difference between a definite, and a corrected integral; for from the expression $\frac{\pi a^2}{4}$, it is impossible to discover fully how x entered into the function from which it was deduced; and, as we shall see, it might have arisen from integrating several other functions; whereas the indefinite, but corrected integral,

$$\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a},$$

by differentiation re-produces the differential coefficient $\sqrt{a^2 - x^2}$, and by its form merely implies that $x = 0$ is its origin, and is still available to find the expression for any area contained between parallel ordinates of which *BC* is one; thus, if we suppose $x = \frac{a}{2}$,

$$\text{area} = \frac{a}{4} \frac{a\sqrt{3}}{2} + \frac{a^2}{2} \frac{\pi}{6} = \frac{a^2}{12} \left(\frac{3\sqrt{3}}{2} + \pi\right).$$

98. When the general value of an integral can be obtained, its value between any required limits can of course be deduced. But there are many integrals whose values between particular limits can be obtained in finite terms, although not their general values by any known method. We shall first give some instances of deducing the value of the definite integral from the indefinite one, and afterwards exemplify the other case.

We may here mention that, in definite integrals, the sign of the result is changed by changing the order of the limits; for if

$$\left(\int_{x=b} - \int_{x=a}\right) u = \phi(a, b), \quad \text{then} \quad \left(\int_{x=a} - \int_{x=b}\right) u = -\phi(a, b).$$

Ex. 1. To find the value of $\int_x x^{m-1}$ between the limits $x = 0$, $x = 1$.

The general integral is $\int_x x^{m-1} = \frac{x^m}{m} + C$;

therefore, making successively $x = 1$, $x = 0$, and subtracting the results,

$$\left(\int_{x=1} - \int_{x=0}\right) x^{m-1} = \frac{1}{m}.$$

Ex. 2. To find the value of $\int_x e^{-ax} \sin mx$ from $x = 0$ to $x = \infty$.

The general integral is

$$\int_x e^{-ax} \sin mx = C - e^{-ax} \frac{a \sin mx + m \cos mx}{m^2 + a^2};$$

therefore, making $x = \infty$, $x = 0$, and subtracting the results,

$$\left(\int_{x=\infty} - \int_{x=0}\right) e^{-ax} \sin mx = \frac{m}{m^2 + a^2}.$$

Ex. 3. To find the value of $\int_x (\sin x)^n$ between the limits $x = 0$, $x = \frac{1}{2}\pi$, n being a positive integer.

The formula of reduction for $\int_x (\sin x)^n$ is

$$\int_x (\sin x)^n = -\frac{1}{n} \cos x \, (\sin x)^{n-1} + \frac{n-1}{n} \int_x (\sin x)^{n-2},$$

in which, if we make successively $x = \frac{1}{2}\pi$, $x = 0$, and subtract the results, since the integrated part vanishes by both substitutions, we find

$$\left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right) (\sin x)^n = \frac{n-1}{n} \left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right) (\sin x)^{n-2};$$

change n into $n - 2$, $n - 4$, &c. successively,

$$\therefore\ \left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)(\sin x)^{n-2} = \frac{n-3}{n-2}\left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)(\sin x)^{n-4}, \text{ \&c.,}$$

till, if n be even, we come to

$$\left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)(\sin x)^{2} = \tfrac{1}{2}\left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right) 1 = \tfrac{1}{2}\cdot\frac{\pi}{2};$$

therefore, multiplying all these equations together and striking out the factors common to both sides, we have, n being even,

$$\left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)(\sin x)^{n} = \frac{(n-1)(n-3)\ldots3.1}{n(n-2)\ldots4.2}\,\frac{\pi}{2}.$$

If n be odd, the last integral will be

$$\left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)(\sin x)^{3} = \frac{2}{3}\left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)\sin x = \frac{2}{3};$$

$$\therefore\ \left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)(\sin x)^{n} = \frac{(n-1)(n-3)\ldots4\,.\,2}{n(n-2)\ldots5\,.\,3}.$$

These results ought to be retained in the memory, as

$$\left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)(\sin x)^{n}$$

is a definite integral which is often met with, and one to which several others may be conveniently reduced.

Thus $\int_x \underline{(\cos x)^n} = -\int_z \underline{(\sin z)^n}$, making $x = \frac{\pi}{2} - z$;

$$\therefore\ \left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)(\cos x)^{n} = -\left(\int_{z=0} - \int_{z=\frac{\pi}{2}}\right)(\sin z)^{n} = \left(\int_{z=\frac{\pi}{2}} - \int_{z=0}\right)(\sin z)^{n},$$

since when $x = 0$, $z = \frac{\pi}{2}$, and when $x = \frac{1}{2}\pi$, $z = 0$;

that is, the value of $\int_x (\cos x)^n$ between the limits $x=0$, $x = \frac{\pi}{2}$, is the same as that of $\int_x (\sin x)^n$ between the same limits.

$\sqrt{a^2-x^2}$ let $P = x^{n-1}\sqrt{a^2-x^2}$

$$\therefore d_x P = \frac{x^{n-2}}{\sqrt{a^2-x^2}}\left(\overline{n-1}\cdot(a^2-x^2) - x^2\right)$$

$$= \frac{x^{n-2}}{\sqrt{a^2-x^2}}\left(\overline{n-1}\cdot a^2 - n x^2\right)$$

$$\therefore \int_x \frac{x^n}{\sqrt{a^2-x^2}} = -\frac{x^{n-1}\sqrt{a^2-x^2}}{n} + \frac{\overline{n-1}a^2}{n}\int_x \frac{x^{n-2}}{\sqrt{a^2-x^2}}$$

$$\therefore \int_{x=0}^{x=a} \frac{x^n}{\sqrt{a^2-x^2}} = \frac{n-1}{n}a^2\int_{x=0}^{x=a} \frac{x^{n-2}}{\sqrt{a^2-x^2}}$$

$$\int_{x=0}^{x=a} \frac{x^{n-2}}{\sqrt{a^2-x^2}} = \frac{n-3}{n-2}a^2\int_{x=0}^{x=a} \frac{x^{n-4}}{\sqrt{a^2-x^2}}$$

till if n be even we come to

$$\int_{x=0}^{x=a} \frac{x^2}{\sqrt{a^2-x^2}} = \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

Multiply up and striking out com.

$$\int_a \frac{x^n}{\sqrt{a^2-x^2}} = \frac{(n-1)(n-3)\cdots\cdots 3.1}{n(n-2)\cdots\cdots 4\cdot 2}a^{2m}\frac{\pi}{2}$$

if n be odd we come to

$$\int_{x=0}^{x=a} \frac{x^3}{\sqrt{a^2-x^2}} = \frac{2}{3}a^2\int_{x=0}^{x=a} \frac{x}{\sqrt{a^2-x^2}}$$

$$= \frac{2}{3}a^2\sqrt{a^2}$$

In ~~this~~ case $= \frac{2a^3}{3}$

$$\int_{x=0}^{x=a} \frac{x^n}{\sqrt{a^2-x^2}} = \frac{(n-1)(n-3)\cdots\cdots 4.2}{n(n-2)\cdots\cdots 5.3}a^{2n+1}$$

Do all these by refor
reduction

Ex. 4. To find the value of $\int_x \frac{x^n}{\sqrt{a^2-x^2}}$ between the limits $x = 0$, $x = a$, n being a positive integer.

This definite integral might be obtained by means of its formula of reduction, as in Ex. 3.; but it may be reduced at once to that integral by putting $\frac{x}{a} = \sin z$, when it becomes

$$a^n \left(\int_{z=\frac{\pi}{2}} - \int_{z=0}\right) (\sin z)^n,$$

since $x = 0$ when $z = 0$, and $x = a$ when $z = \frac{\pi}{2}$. Hence changing n into $2r + 1$, and $2r$, according as it is odd or even,

$$\left(\int_{x=a} - \int_{x=0}\right) \frac{x^{2r+1}}{\sqrt{a^2 - x^2}} = a^{2r+1} \frac{2r(2r-2)\ldots4.2}{(2r+1)(2r-1)\ldots5.3},$$

$$\left(\int_{x=a} - \int_{x=0}\right) \frac{x^{2r}}{\sqrt{a^2 - x^2}} = a^{2r} \frac{(2r-1)(2r-3)\ldots3.1}{2r(2r-2)\ldots4.2} \frac{\pi}{2}.$$

Ex. 5. To find the value of $\int_x \frac{1}{(a^2 + x^2)^n}$ between the limits $x = 0$, $x = \infty$, n being a positive integer.

See p
y 2

Putting $\frac{x}{a} = \cot z$, this becomes

$$-\frac{1}{a^{2n-1}} \left(\int_{z=0} - \int_{z=\frac{\pi}{2}}\right) (\sin z)^{2n-2},$$

since $x = 0$, $z = \frac{\pi}{2}$, $x = \infty$, $z = 0$ are corresponding values;

$$\therefore \left(\int_{x=\infty} - \int_{x=0}\right) \frac{1}{(a^2 + x^2)^n} = \frac{1}{a^{2n-1}} \frac{(2n-3)(2n-5)\ldots1}{(2n-2)(2n-4)\ldots2} \cdot \frac{\pi}{2}.$$

Similarly, by putting $\frac{x}{a} = \text{vers}\, z$, we find

$$\left(\int_{x=a} - \int_{x=0}\right)(2ax - x^2)^{n+\frac{1}{2}} = a^{2n+2}\,\frac{(2n+1)(2n-1)\dots1}{(2n+2)(2n)\quad\dots2}\,\frac{\pi}{2}.$$

Ex. 6. To find the value of $\int_x (\sin x)^m (\cos x)^n$ between the limits $x = 0$, $x = \frac{\pi}{2}$; m and n being even integers.

By the formula of reduction Ex. 2. Art. 75, we have

$$\left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)(\sin x)^m (\cos x)^n = \frac{n-1}{m+n}\left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)(\sin x)^m (\cos x)^{n-2},$$

&c. = &c.,

changing successively n into $n-2$, $n-4$, &c.; till at last we come to

$$\left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)(\sin x)^m (\cos x)^2 = \frac{1}{m+2}\left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)(\sin x)^m;$$

therefore, multiplying all these equations together,

$$\left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)(\sin x)^m (\cos x)^n = \frac{(n-1)(n-3)\dots3.1}{(m+n)(m+n-2)\dots(m+2)}$$

$$\times \left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)(\sin x)^m$$

$$= \frac{(n-1)(n-3)\dots3.1}{(m+n)(m+n-2)\dots(m+2)}\cdot\frac{(m-1)(m-3)\dots3.1}{m(m-2)\dots4.2}\,\frac{\pi}{2}$$

$$= \frac{(n-1)(n-3)\dots3.1 \times (m-1)(m-3)\dots3.1}{(m+n)(m+n-2)\dots2.1}\,\frac{\pi}{2}.$$

Ex. 7. To find the value of $\left(\int_{x=1} - \int_{x=0}\right) x^m \left(\log_e \frac{1}{x}\right)^n$, m and n being any positive quantities.

Integrating by parts, the formula of reduction is

$$\int_x x^m \left(\log_\epsilon \frac{1}{x}\right)^n = \frac{x^{m+1}}{m+1}\left(\log_\epsilon \frac{1}{x}\right)^n + \frac{n}{m+1}\int_x x^m \left(\log_\epsilon \frac{1}{x}\right)^{n-1};$$

now the integrated part, which depends upon

$$\left(\log_\epsilon \frac{1}{x}\right)^n \div \left(\frac{1}{x}\right)^{m+1}$$

$$\text{or } \left(\log_\epsilon \frac{1}{x}\right)^n \div \left\{1 + (m+1)\log_\epsilon \frac{1}{x} + \&\text{c.}\right.$$

$$\left. + \frac{(m+1)^{n+1}}{1 \, . \, 2 \, . \, 3 \ldots (n+1)}\left(\log_\epsilon \frac{1}{x}\right)^{n+1} + \&\text{c.}\right\},$$

manifestly vanishes at both limits,

$$\therefore \left(\int_{x=1} - \int_{x=0}\right) x^m \left(\log_\epsilon \frac{1}{x}\right)^n = \frac{n}{m+1}\left(\int_{x=1} - \int_{x=0}\right) x^m \left(\log_\epsilon \frac{1}{x}\right)^{n-1}.$$

By this formula when n is a fraction, the proposed integral can be made to depend upon

$$\left(\int_{x=1} - \int_{x=0}\right) x^m \left(\log_\epsilon \frac{1}{x}\right)^{r-1} \text{ (where } r \text{ is } > 0 < 1),$$

$$\text{or } \frac{1}{(m+1)^r}\left(\int_{z=1} - \int_{z=0}\right)\left(\log_\epsilon \frac{1}{z}\right)^{r-1} \text{ (making } x^{m+1} = z),$$

the value of which we shall hereafter shew how to find.

But when n is an integer,

$$\left(\int_{x=1} - \int_{x=0}\right) x^m \left(\log_\epsilon \frac{1}{x}\right)^n = \frac{n(n-1)\ldots 3.2.1}{(m+1)^{n+1}}.$$

Hence also, multiplying both sides by $(-1)^n$,

$$\left(\int_{x=1} - \int_{x=0}\right) x^m (\log_\epsilon x)^n = (-1)^n \frac{n(n-1)\ldots 3.2.1}{(m+1)^{n+1}}.$$

Also, since

$$x^x = 1 + x(\log_\epsilon x) + \frac{x^2(\log_\epsilon x)^2}{1.2} + \frac{x^3(\log_\epsilon x)^3}{1.2.3} + \text{\&c.},$$

$$\text{and } \left(\int_{x=1} - \int_{x=0}\right)\frac{x^n(\log_\epsilon x)^n}{1.2.3\ldots n} = (-1)^n\frac{1}{(n+1)^{n+1}},$$

$$\therefore \left(\int_{x=1} - \int_{x=0}\right) x^x = \frac{1}{1^1} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \text{\&c.};$$

$$\text{similarly, } \left(\int_{x=1} - \int_{x=0}\right) x^{-x} = \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \text{\&c.}$$

If in the above two formulæ $m = 0$, we have

$$\left(\int_{x=1} - \int_{x=0}\right)(\log_\epsilon x)^n = (-1)^n\, 1.2.3\ldots n,$$

$$\left(\int_{x=1} - \int_{x=0}\right)\left(\log_\epsilon \frac{1}{x}\right)^n = 1.2.3\ldots n.$$

Ex. 8. To find the value of $\left(\int_{x=\infty} - \int_{x=0}\right) x^n e^{-x}$, n being a positive integer.

Integrating by parts

$$\int_x x^n e^{-x} = -x^n e^{-x} + n\int_x x^{n-1} e^{-x},$$

$$\therefore \left(\int_{x=\infty} - \int_{x=0}\right) x^n e^{-x} = n\left(\int_{x=\infty} - \int_{x=0}\right) x^{n-1} e^{-x},$$

$$\text{\&c.} = \text{\&c.}$$

$$\left(\int_{x=\infty} - \int_{x=0}\right) x e^{-x} = \left(\int_{x=\infty} - \int_{x=0}\right) e^{-x} = 1,$$

therefore, multiplying these equations together,

$$\left(\int_{x=\infty} - \int_{x=0}\right) x^n e^{-x} = 1.2.3\ldots n.$$

99. The following properties of definite integrals, where the values of the quantity under the sign of definite integration are periodical, it will be useful to notice.

If $f(x)$ do not become infinite between $x=0$ and $x=a$, and be such that $f(x)=-f(a-x)$, its integral between the limits $x=0$, $x=a$, will vanish.

For if $f(x)$ be considered as the ordinate of a curve, this curve will intersect the axis when $x=\frac{1}{2}a$; because, making $x=\frac{1}{2}a$, we have $2f(\frac{1}{2}a)=0$, or $f(\frac{1}{2}a)=0$. Also between the ordinates corresponding to $x=0$, $x=a$, the curve will consist of two equal and similar portions, one above, and the other below the axis; since making $x=\frac{1}{2}a+z$, we have

$$f(\tfrac{1}{2}a+z)=-f(\tfrac{1}{2}a-z),$$

or the ordinates equal, at equal distances from the point of intersection, but of different signs. But the area corresponding to a negative ordinate is negative; therefore the sum of the areas of the two portions, that is, the required integral is nothing.

Hence the value of the definite integral

$$(\textstyle\int_{x=\pi}-\int_{x=0})\cos x\,.\,R\,\{(\cos x)^2,\ \sin x\},$$

which occurs in the investigation of the attraction of spheroids, is nothing.

Again, if $f(x)$ be such that $f(x)=f(a-x)$, and consequently $f(\frac{1}{2}a+z)=f(\frac{1}{2}a-z)$, the area between the ordinates corresponding to $x=0$, $x=\frac{1}{2}a$, will be equal to that between the ordinates corresponding to $x=\frac{1}{2}a$, $x=a$, and will have the same sign,

$$\therefore\ (\textstyle\int_{x=a}-\int_{x=0})f(x)=2(\int_{x=\frac{a}{2}}-\int_{x=0})f(x).$$

If $f(x)$ be such that $f(x)=f(a+x)$, then since for every addition of a to the abscissa, the same area will recur,

$$(\textstyle\int_{x=na}-\int_{x=0})f(x)=n\,(\int_{x=a}-\int_{x=0})\,f(x).$$

100. Instead of taking an integral between the limits a and b, we may take it between the limits a and c and between the limits c and b, (c being any intermediate value of x) and add the results together, and we shall obtain the same value for the definite integral in either case; for considering the expression to be integrated as representing the ordinate of a curve, the first area will be equal to the sum of the two others. In the same manner we may resolve an integral which is to be taken between given limits, into any number of others, the limits of which are intermediate to those of the former, the termination of one integral being the origin of the next, it being understood that the expression to be integrated does not become infinite for any value of x between the extreme limits.

101. We next come to the case of those integrals whose values between particular limits can be obtained in finite terms, although not their general values by any known method. The following are the principal methods of finding the values of this sort of definite integrals.

1. By combining the values of other definite integrals.
2. By expanding the expression, integrating, and summing the resulting series.
3. By using impossible quantities.
4. By differentiating or integrating under the sign of definite integration.

We shall give a few of the most remarkable results that have been obtained by each of these methods.

102. To shew that $(\int_{t=\infty} - \int_{t=0}) e^{-t^2} = \frac{1}{2}\sqrt{\pi}$.

By combining the definite integrals Ex. 3, Art. 98, we have

$$(\int_{x=1} - \int_{x=0}) \frac{x^{2r}}{\sqrt{1-x^2}} \times (\int_{x=1} - \int_{x=0}) \frac{x^{2r+1}}{\sqrt{1-x^2}}$$

$$= \frac{1}{2r+1} \cdot \frac{\pi}{2} \text{ (r being an integer).} \quad (1).$$

Since x is not restricted to represent a simple quantity, let $x = e^{-\frac{t^2}{2r}}$,

$$\therefore \int_x \frac{x^{2r}}{\sqrt{1-x^2}} = -\frac{1}{\sqrt{r}} \int_t \frac{te^{-t^2\left(1+\frac{1}{2r}\right)}}{\sqrt{\left(1-e^{-\frac{t^2}{r}}\right)r}};$$

but if r be a very large number,

$$e^{-\frac{t^2}{r}} = 1 - \frac{t^2}{r} + \frac{t^4}{2r^2} - \&c. \quad \therefore \sqrt{\left(1-e^{-\frac{t^2}{r}}\right)r} = t\left(1-\frac{t^2}{4r}\right);$$

also when $x = 0$, $t = \infty$, and when $x = 1$, $t = 0$,

$$\therefore \left(\int_{x=1} - \int_{x=0}\right)\frac{x^{2r}}{\sqrt{1-x^2}} = \frac{1}{\sqrt{r}}\left(\int_{t=\infty} - \int_{t=0}\right)\frac{e^{-t^2\left(1+\frac{1}{2r}\right)}}{1-\frac{t^2}{4r}};$$

similarly,

$$\int_x \frac{x^{2r+1}}{\sqrt{1-x^2}} = -\frac{1}{\sqrt{r}}\int_t \frac{te^{-t^2\left(1+\frac{1}{r}\right)}}{\sqrt{\left(1-e^{-\frac{t^2}{r}}\right)r}} = -\frac{1}{\sqrt{r}}\int_t \frac{e^{-t^2\left(1+\frac{1}{r}\right)}}{1-\frac{t^2}{4r}},$$

when r is a very large number;

$$\therefore \left(\int_{x=1} - \int_{x=0}\right)\frac{x^{2r+1}}{\sqrt{1-x^2}} = \frac{1}{\sqrt{r}}\left(\int_{t=\infty} - \int_{t=0}\right)\frac{e^{-t^2\left(1+\frac{1}{r}\right)}}{1-\frac{t^2}{4r}};$$

hence, substituting in equation (1),

$$\left(2+\frac{1}{r}\right)\left(\int_{t=\infty} - \int_{t=0}\right)\frac{e^{-t^2\left(1+\frac{1}{2r}\right)}}{1-\frac{t^2}{4r}} \times \left(\int_{t=\infty} - \int_{t=0}\right)\frac{e^{-t^2\left(1+\frac{1}{r}\right)}}{1-\frac{t^2}{4r}} = \frac{\pi}{2};$$

now make r infinite, $\therefore 2\left\{\left(\int_{t=\infty} - \int_{t=0}\right)e^{-t^2}\right\}^2 = \frac{\pi}{2}$,

or $\left(\int_{t=\infty} - \int_{t=0}\right)e^{-t^2} = \frac{1}{2}\sqrt{\pi}$.

103. This result will enable us to find the values of some other definite integrals.

Thus let $e^{-t^2} = x$, or $t = \left(\log_\epsilon \frac{1}{x}\right)^{\frac{1}{2}}$, also $-2te^{-t^2} d_x t = 1$,

or $e^{-t^2} d_x t = -\frac{1}{2t}$, $\therefore \int_t e^{-t^2} = \int_x e^{-t^2} d_x t = -\frac{1}{2}\int_x \left(\log_\epsilon \frac{1}{x}\right)^{-\frac{1}{2}}$;

also when $t = \infty$, $x = 0$, and when $t = 0$, $x = 1$;

$$\therefore \left(\int_{t=\infty} - \int_{t=0}\right) e^{-t^2} = \frac{1}{2}\left(\int_{x=1} - \int_{x=0}\right) \left(\log_\epsilon \frac{1}{x}\right)^{-\frac{1}{2}} = \frac{\sqrt{\pi}}{2};$$

$$\text{or } \left(\int_{x=1} - \int_{x=0}\right) \left(\log_\epsilon \frac{1}{x}\right)^{-\frac{1}{2}} = \sqrt{\pi}.$$

Hence, Art. 98, Ex. 7,

$$\left(\int_{x=1} - \int_{x=0}\right) \left(\log_\epsilon \frac{1}{x}\right)^{n+\frac{1}{2}} = \frac{(2n+1)(2n-1)\ldots 3.1}{2^{n+1}}\sqrt{\pi};$$

which is also the value of $2\left(\int_{t=\infty} - \int_{t=0}\right) e^{-t^2} t^{2n+2}$.

104. To prove that $\left(\int_{x=\infty} - \int_{x=0}\right) \frac{x^{m-1}}{1+x^n} = \frac{\pi}{n \sin \frac{m\pi}{n}}$,

m and n being positive quantities, and $m < n$.

When m and n are integers, this result may be obtained from the indefinite integral of $\frac{x^{m-1}}{1+x^n}$ already found, Art. 42; but we shall employ the following independent process, which consists in expressing the value of the integral in a series, and summing the series.

Instead of integrating between the limits $x = 0$, $x = \infty$, we shall separately integrate between the limits $x = 0$, $x = 1$, and $x = 1$, $x = \infty$, and add the results.

$$\text{Now } \int_x \frac{x^{m-1}}{1+x^n} = \int_x x^{m-1}\left(1 - x^n + x^{2n} - x^{3n} + \&\text{c.}\right)$$

$$= \frac{x^m}{m} - \frac{x^{m+n}}{m+n} + \frac{x^{m+2n}}{m+2n} - \&\text{c.} + C;$$

$$\therefore \left(\int_{x=1} - \int_{x=0}\right)\frac{x^{m-1}}{1+x^n} = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \&\text{c.}$$

Again,

$$\int_x \frac{x^{m-1}}{1+x^n} = \int_x \frac{x^{m-n-1}}{1+x^{-n}} = \int_x x^{m-n-1}\left(1 - x^{-n} + x^{-2n} - x^{-3n} + \&\text{c.}\right)$$

$$= \frac{x^{m-n}}{m-n} - \frac{x^{m-2n}}{m-2n} + \frac{x^{m-3n}}{m-3n} - \&\text{c.} + C,$$

$$\therefore \left(\int_{x=\infty} - \int_{x=1}\right)\frac{x^{m-1}}{1+x^n} = \frac{1}{n-m} - \frac{1}{2n-m} + \frac{1}{3n-m} - \&\text{c.}$$

(since n is greater than m); therefore by addition

$$\left(\int_{x=\infty} - \int_{x=0}\right)\frac{x^{m-1}}{1+x^n} = \left(\frac{1}{m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{4n-m} + \&\text{c.}\right)$$

$$+ \left(\frac{1}{n-m} - \frac{1}{n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} + \&\text{c.}\right) \quad (1),$$

and it remains to sum these series.

$$\text{Now } \cos x = \left\{1 - \left(\frac{2x}{\pi}\right)^2\right\}\left\{1 - \left(\frac{2x}{3\pi}\right)^2\right\} \&\text{c.}; \text{ make } \frac{2x}{\pi} = \theta,$$

$$\therefore \cos\frac{\pi\theta}{2} = (1 - \theta^2)\left\{1 - \left(\frac{\theta}{3}\right)^2\right\}\left\{1 - \left(\frac{\theta}{5}\right)^2\right\}.\ \&\text{c.}$$

$$= (1-\theta)(1+\theta)\left(1 - \frac{\theta}{3}\right)\left(1 + \frac{\theta}{3}\right).\ \&\text{c.}$$

take the differential coefficient with respect to θ of the logarithm of each side,

$$\therefore \ \frac{\pi}{2}\tan\frac{\pi\theta}{2} = \frac{1}{1-\theta} - \frac{1}{1+\theta} + \frac{1}{3-\theta} - \frac{1}{3+\theta} + \frac{1}{5-\theta} - \&c.$$

divide this equation by (n), and then make $n\theta = m$,

$$\therefore \ \frac{\pi}{2n}\tan\frac{m\pi}{2n} = \frac{1}{n-m} - \frac{1}{n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} + \&c.$$

Again, since

$$\sin x = x\left\{1 - \left(\frac{x}{\pi}\right)^2\right\}\left\{1 - \left(\frac{x}{2\pi}\right)^2\right\}\&c.; \text{ making } x = \pi\theta,$$

$$\text{we have } \sin\pi\theta = \pi\theta(1-\theta^2)\left\{1 - \left(\frac{\theta}{2}\right)^2\right\}\left\{1 - \left(\frac{\theta}{3}\right)^2\right\}.\ \&c.$$

$$= \pi\theta(1-\theta)(1+\theta)\left(1 - \frac{\theta}{2}\right)\left(1 + \frac{\theta}{2}\right).\ \&c.$$

take the differential coefficient with respect to θ of the logarithm of each side,

$$\therefore \ \pi\cot\pi\theta = \frac{1}{\theta} - \frac{1}{1-\theta} + \frac{1}{1+\theta} - \frac{1}{2-\theta} + \frac{1}{2+\theta} - \&c.;$$

divide this equation by $2n$, and then make $2n\theta = m$,

$$\therefore \ \frac{\pi}{2n}\cot\frac{m\pi}{2n} = \frac{1}{m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{4n-m} + \frac{1}{4n+m} - \&c.$$

Hence, substituting in equation (1),

$$\left(\int_{x=\infty} - \int_{x=0}\right)\frac{x^{m-1}}{1+x^n} = \frac{\pi}{2n}\left\{\cot\frac{m\pi}{2n} + \tan\frac{m\pi}{2n}\right\} = \frac{\pi}{n\sin\frac{m\pi}{n}}.$$

105. Similarly, by expanding and integrating we find

$$\left(\int_{x=1} - \int_{x=0}\right)\frac{x^{m-1} - x^{n-m-1}}{1-x^n}$$

$$= \left(\int_{x=1} - \int_{x=0}\right)\left(x^{m-1} - x^{n-m-1}\right)\left(1 + x^n + x^{2n} + \&c.\right)$$

$$= \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \&c.$$

$$= \frac{\pi}{2n}\left(\cot\frac{m\pi}{2n} - \tan\frac{m\pi}{2n}\right) = \frac{\pi}{n\tan\dfrac{m\pi}{n}}.$$

Also $\left(\int_{x=1} - \int_{x=0}\right)\dfrac{x^{m-1} + x^{n-m-1}}{1 + x^n} = \dfrac{\pi}{n\sin\dfrac{m\pi}{n}}.$

106. We shall further apply this method to find the value of

$$\left(\int_{x=\infty} - \int_{x=0}\right)\frac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}}\sin mx, \quad a \text{ being } < \pi.$$

$$\text{Since } \frac{1}{e^{\pi x} - e^{-\pi x}} = e^{-\pi x} + e^{-3\pi x} + e^{-5\pi x} + \&c.,$$

the proposed integral becomes

$$\left(\int_{x=\infty} - \int_{x=0}\right)\left(e^{-(\pi-a)x} + e^{-(3\pi-a)x} + \&c. + e^{-(\pi+a)x} + e^{-(3\pi+a)x} + \&c.\right)\sin mx,$$

which (Ex. 2. Art. 98) $= \dfrac{m}{(\pi-a)^2 + m^2} + \dfrac{m}{(3\pi-a)^2 + m^2} + \&c.$

$$+ \frac{m}{(\pi+a)^2 + m^2} + \frac{m}{(3\pi+a)^2 + m^2} + \&c.$$

But $e^m + 2\cos a + e^{-m} = 4\left(\sin\dfrac{a}{2}\right)^2\left\{1 + \left(\dfrac{m}{\pi-a}\right)^2\right\}\left\{1 + \left(\dfrac{m}{\pi+a}\right)^2\right\}$

$$\left\{1 + \left(\frac{m}{3\pi-a}\right)^2\right\}\left\{1 + \left(\frac{m}{3\pi+a}\right)^2\right\}\&c.;$$

therefore, taking the differential coefficient relative to m of the logarithm of each member,

$$\frac{e^m - e^{-m}}{e^m + 2\cos a + e^{-m}} = \frac{2m}{(\pi-a)^2 + m^2} + \frac{2m}{(\pi+a)^2 + m^2} + \frac{2m}{(3\pi-a)^2 + m^2} + \&c.$$

$$\therefore \left(\int_{x=\infty} - \int_{x=0}\right)\frac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}}\sin mx = \frac{1}{2}\,\frac{e^m - e^{-m}}{e^m + 2\cos a + e^{-m}}.$$

107. We shall next consider the definite integral

$$\left(\int_{x=1} - \int_{x=0}\right) \frac{x^{p-1}}{(1-x^n)^{\frac{n-q}{n}}}\,\cdot$$

which is known as the first of Euler's Integrals.

Its value will be a function of p, q, n, which are supposed to be integers; and this value we shall denote by $\left\{\frac{p}{q}\right\}$, because in the following reduction n, the index of the variable under the vinculum, remains unaltered.

When $p + q = n$, the value can be immediately obtained;

for $\displaystyle\int_x \frac{x^{p-1}}{(1-x^n)^{\frac{p}{n}}} = \int_x \frac{1}{x(x^{-n}-1)^{\frac{p}{n}}} = \int_z \frac{z^{p-1}}{1+z^n}$, if $x^{-n} - 1 = z^{-n}$;

therefore, since $\left.\begin{matrix} x = 1, & x = 0, \\ z = \infty, & z = 0, \end{matrix}\right\}$ are corresponding values,

$$\left\{\int_{x=1} - \int_{x=0}\right\} \frac{x^{p-1}}{(1-x^n)^{\frac{p}{n}}} = \left(\int_{z=\infty} - \int_{z=0}\right) \frac{z^{p-1}}{1+z^n} = \frac{\pi}{n \sin \frac{p\pi}{n}}\,.$$

There is yet another case (viz. when $q = n$) in which we can find the value of $\left\{\frac{p}{q}\right\}$ immediately, for

$$\left\{\frac{p}{n}\right\} = \left(\int_{x=1} - \int_{x=0}\right) x^{p-1} = \frac{1}{p}\,.$$

We shall now prove that $\left\{\frac{p}{q}\right\}$ can in all cases be made to depend upon $\left\{\frac{p_{\prime}}{q_{\prime}}\right\}$ where $p_{\prime}$ and $q_{\prime}$ are each $< n$.

If we assume $x^n = 1 - z^n$,

we find $\displaystyle\int_x \frac{x^{p-1}}{(1-x^n)^{\frac{n-q}{n}}} = -\int_z \frac{z^{q-1}}{(1-z^n)^{\frac{n-p}{n}}}$, therefore, since $\left.\begin{matrix} x = 1, & x = 0, \\ z = 0, & z = 1, \end{matrix}\right\}$ are corresponding values,

$$\left(\int_{x=1} - \int_{x=0}\right) \frac{x^{p-1}}{(1-x^n)^{\frac{n-q}{n}}} = \left(\int_{z=1} - \int_{z=0}\right) \frac{z^{q-1}}{(1-z^n)^{\frac{n-p}{n}}},$$

$$\text{or } \left\{\frac{p}{q}\right\} = \left\{\frac{q}{p}\right\},$$

that is, p and q may be interchanged in the expression to be integrated without altering the value of the definite integral.

Again, to make $\int_x \frac{x^{p-1}}{(1-x^n)^{\frac{n-q}{n}}}$ depend upon $\int_x \frac{x^{p-n-1}}{(1-x^n)^{\frac{n-q}{n}}}$,

assume $P = x^{p-n}(1-x^n)^{\frac{q}{n}}$, according to Art. 55,

$$\therefore \int_x \frac{x^{p-1}}{(1-x^n)^{\frac{n-q}{n}}} = -\frac{P}{p+q-n} + \frac{p-n}{p+q-n} \int_x \frac{x^{p-n-1}}{(1-x^n)^{\frac{n-q}{n}}},$$

consequently, since P vanishes at both limits, q being positive and $p > n$,

$$\left(\int_{x=1} - \int_{x=0}\right) \frac{x^{p-1}}{(1-x^n)^{\frac{n-q}{n}}} = \frac{p-n}{p+q-n} \left(\int_{x=1} - \int_{x=0}\right) \frac{x^{p-n-1}}{(1-x^n)^{\frac{n-q}{n}}},$$

$$\text{or } \left\{\frac{p}{q}\right\} = \frac{p-n}{p+q-n} \left\{\frac{p-n}{q}\right\}.$$

By this formula, if $p > n$, $\left\{\frac{p}{q}\right\}$ can be made to depend upon $\left\{\frac{p_{,}}{q}\right\}$ where $p_{,} < n$; and if $q > n$, in a similar manner $\left\{\frac{p_{,}}{q}\right\} = \left\{\frac{q}{p_{,}}\right\}$ can be made to depend upon $\left\{\frac{q_{,}}{p'}\right\} = \left\{\frac{p_{,}}{q_{,}}\right\}$ where $q_{,}$ is $< n$.

The continuation of this investigation would shew that every case of $\left\{\frac{p}{q}\right\}$ can be made to depend upon the auxiliary integrals

$$\left\{\frac{n-2}{1}\right\},\ \left\{\frac{n-3}{2}\right\},\ \left\{\frac{n-4}{3}\right\},\ \&c.,$$

in number $\frac{n-2}{2}$ or $\frac{n-1}{2}$ according as n is even or odd.

108. We come next to the consideration of the second of Euler's Integrals,

$$\Gamma(n) = (\textstyle\int_{x=1} - \int_{x=0}) \left(\log_\epsilon \frac{1}{x}\right)^{n-1}.$$

When n is an integer we have seen (Ex. 7. Art. 98.), that

$$(\textstyle\int_{x=1} - \int_{x=0}) \left(\log_\epsilon \frac{1}{x}\right)^{n-1} = (n-1)(n-2)\ldots 2 . 1;$$

in this case then, $\Gamma(n)$ denotes the product of all whole numbers $<n$: when n is a fraction, the indefinite product $(n-1)(n-2)\ldots$ is represented by the transcendent

$$(\textstyle\int_{x=1} - \int_{x=0}) \left(\log_\epsilon \frac{1}{x}\right)^{n-1},$$

so that $\Gamma(n)$ furnishes the means of generalizing formulæ involving $(n-1)(n-2)\ldots$ when n is an integer, so as to hold for all values of n; this makes the investigation of its properties of importance; the following are the principal ones.

In the first place, since for all values of n

$$(\textstyle\int_{x=1} - \int_{x=0}) \left(\log_\epsilon \frac{1}{x}\right)^{n} = n (\textstyle\int_{x=1} - \int_{x=0}) \left(\log_\epsilon \frac{1}{x}\right)^{n-1},$$

$$\text{we have, } \Gamma(n+1) = n\Gamma(n),$$

which is the fundamental property of the function Γ, and shews that if its values corresponding to all values of n included between two consecutive integers were computed, then all other values could be deduced.

Next, let n be a whole number, and m any positive quantity, then (Art. 55, Ex. 7),

$$\int_x x^{m-1}(1-x)^{n-1}=\frac{x^m(1-x)^{n-1}}{m+n-1}+\frac{n-1}{m+n-1}\int_x x^{m-1}(1-x)^{n-2};$$

$$\therefore\ \left(\int_{x=1}-\int_{x=0}\right)x^{m-1}(1-x)^{n-1}=\frac{n-1}{m+n-1}\left(\int_{x=1}-\int_{x=0}\right)x^{m-1}(1-x)^{n-2},$$

$$=\frac{(n-1)(n-2)\ldots2.1}{(m+n-1)(m+n-2)\ldots(m+1)}\cdot\frac{1}{m}.$$

Now suppose n to be fractional, then the numerator becomes the indefinite product $(n-1)(n-2)\ldots=\Gamma(n)$; and the denominator

$$(m+n-1)(m+n-2)\ldots(m+1)\,m=\frac{\Gamma(m+n)}{\Gamma(m)}$$

by the fundamental formula,

$$\therefore\ \left(\int_{x=1}-\int_{x=0}\right)x^{m-1}(1-x)^{n-1}=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

Let $m+n=1$ or $n-1=-m$; then $\Gamma(m+n)=\Gamma(1)=1$, and the first member becomes

$$\left(\int_{x=1}-\int_{x=0}\right)\frac{x^{m-1}}{(1-x)^m}=\frac{\pi}{\sin m\pi}\ \text{(Art. 107)};$$

$$\therefore\ \frac{\pi}{\sin m\pi}=\Gamma(m).\Gamma(1-m)\ \ (m \text{ being } >0<1).$$

Hence it is sufficient to know $\Gamma(m)$ from $m=0$ to $m=\frac{1}{2}$, in order to deduce all the other values.

Let $m=\frac{1}{2}$, then $\sqrt{\pi}=\Gamma(\frac{1}{2})$, as found in Art. 103.

109. As an instance of the employment of impossible quantities in finding definite integrals, we shall take

$$\left(\int_{x=\infty}-\int_{x=0}\right)x^p e^{-qx}\cos rx,$$

p being a positive integer, and q, r, any positive quantities.

Putting for the cosine its exponential value,

$$\int_x x^p e^{-qx} \cos rx = \frac{1}{2}\int_x x^p e^{-(q-r\sqrt{-1})x} + \frac{1}{2}\int_x x^p e^{-(q+r\sqrt{-1})x}.$$

Let $(q - r\sqrt{-1})x = t$, so that when $x = 0$, $t = 0$, and when $x = \infty$, $t = \infty$; then the first of the above integrals becomes

$$\frac{1}{2}\frac{1}{(q-r\sqrt{-1})^{p+1}} \cdot \int_t t^p e^{-t}$$

which (between the limits $t = 0$, $t = \infty$) $= \dfrac{1 \,.\, 2 \,.\, 3 \ldots p}{2(q - r\sqrt{-1})^{p+1}}$;

similarly, changing the sign of r, the second becomes

$$= \frac{1 \,.\, 2 \,.\, 3 \ldots p}{2(q + r\sqrt{-1})^{p+1}}.$$

The sum of these is

$$\frac{1 \,.\, 2 \,.\, 3 \ldots p}{2} \, \frac{(q + r\sqrt{-1})^{p+1} + (q - r\sqrt{-1})^{p+1}}{(q^2 + r^2)^{p+1}}$$

$$= 1 \,.\, 2 \,.\, 3 \ldots p \frac{\cos(p+1)\alpha}{(q^2 + r^2)^{\frac{p+1}{2}}}, \text{ if } \tan \alpha = \frac{r}{q},$$

which is the value of $\left(\int_{x=\infty} - \int_{x=0}\right) x^p e^{-qx} \cos rx$.

Similarly, we may shew that

$$\left(\int_{x=\infty} - \int_{x=0}\right) x^p e^{-qx} \sin rx = 1 \,.\, 2 \,.\, 3 \ldots p \frac{\sin(p+1)\alpha}{(q^2 + r^2)^{\frac{p+1}{2}}}.$$

110. The method of differentiating or integrating under the sign of definite integration is one of great utility; it is founded upon the truth of the equation

$$d_c\left\{\left(\int_{x=b} - \int_{x=a}\right)u\right\} = \left(\int_{x=b} - \int_{x=a}\right)(d_c u),$$

c being any constant found in u.

That we here differentiate with regard to a constant ought to create no difficulty; for the series of operations by which we obtain the differential coefficient of a function with respect to any quantity entering into it, can be performed equally well whether that quantity be essentially variable, or we only suppose a variation where none really exists: we have indeed only to regard u as a function of two independent quantities x and c.

$$\text{Now let } \textstyle\int_x u = \phi(x, c),$$

$$\therefore\ d_c\phi(x, c) = d_c(\textstyle\int_x u);$$

$$\text{but } d_x\phi(x, c) = u,$$

$$\therefore\ d_c d_x\phi(x, c) = d_c u, \text{ or } d_x d_c\phi(x, c) = d_c u;$$

$$\therefore\ d_c\phi(x, c) = \textstyle\int_x (d_c u);$$

$$\therefore \text{ equating these values of } d_c\phi(x, c),$$

$$d_c(\textstyle\int_x u) = \textstyle\int_x (d_c u).$$

Now as x and c are independent of one another, this equation is true for all values of x, and therefore when $x = b$, $x = a$;

$$\therefore\ d_c(\textstyle\int_{x=b} u) = \textstyle\int_{x=b}(d_c u),$$

$$d_c(\textstyle\int_{x=a} u) = \textstyle\int_{x=a}(d_c u);$$

$$\therefore \text{ by subtraction, } d_c\{(\textstyle\int_{x=b} - \textstyle\int_{x=a})\, u\} = \{\textstyle\int_{x=b} - \textstyle\int_{x=a}\}(d_c u).$$

Hence,

$$d_c^2\{(\textstyle\int_{x=b} - \textstyle\int_{x=a}) u\} = d_c\{(\textstyle\int_{x=b} - \textstyle\int_{x=a})(d_c u)\} = (\textstyle\int_{x=b} - \textstyle\int_{x=a})(d_c^2 u),$$

and so on for any number of differentiations.

Similarly, it may be shewn that

$$\textstyle\int_c^n\{(\textstyle\int_{x=b} - \textstyle\int_{x=a})\, u\} = \{\textstyle\int_{x=b} - \textstyle\int_{x=a}\}(\textstyle\int_c^n u).$$

111. Our first application of this principle shall be, starting from known results, to deduce the values of certain definite integrals.

Ex. 1. $\left(\int_{x=\infty} - \int_{x=0}\right) \dfrac{1}{x^2 + c} = \dfrac{1}{\sqrt{c}} \dfrac{\pi}{2}$;

$$\therefore \left(\int_{x=\infty} - \int_{x=0}\right) d_c^n \left(\frac{1}{x^2 + c}\right) = \frac{\pi}{2} d_c^n \left(\frac{1}{\sqrt{c}}\right),$$

or $\left(\int_{x=\infty} - \int_{x=0}\right) \dfrac{1}{(x^2 + c)^{n+1}} = c^{-\frac{2n+1}{2}} \dfrac{1 \,.\, 3 \,.\, 5 \ldots (2n - 1)}{2 \,.\, 4 \,.\, 6 \ldots 2n} \dfrac{\pi}{2}$,

which, putting a^2 instead of c, agrees with Ex. 5. Art. 98.

Ex. 2. $\left(\int_{x=1} - \int_{x=0}\right) \dfrac{x^{m-1} - x^{n-m-1}}{1 - x^n} = \dfrac{\pi}{n} \cot \dfrac{m\pi}{n}$ (Art. 105);

differentiate with respect to m,

$$\therefore \left(\int_{x=1} - \int_{x=0}\right) \frac{x^{m-1} + x^{n-m-1}}{1 - x^n} \,.\, \log_\epsilon \frac{1}{x} = \left(\frac{\pi}{n}\right)^2 \left(\operatorname{cosec} \frac{m\pi}{n}\right)^2.$$

Ex. 3. $\left(\int_{x=1} - \int_{x=0}\right) x^{m-1} = \dfrac{1}{m}$;

integrate with respect to m,

$$\left(\int_{x=1} - \int_{x=0}\right) \left(\frac{x^{m-1}}{\log_\epsilon x} + C\right) = \log_\epsilon m.$$

To find C, let $m = 1$, $\therefore \left(\int_{x=1} - \int_{x=0}\right) \left(\dfrac{1}{\log_\epsilon x} + C\right) = 0$,

$$\therefore \left(\int_{x=1} - \int_{x=0}\right) \frac{x^{m-1} - 1}{\log_\epsilon x} = \log_\epsilon m;$$

which can be verified by putting

$$1 + (m - 1) \log_\epsilon x + \frac{(m - 1)^2}{1 \,.\, 2} (\log_\epsilon x)^2 + \&c. \text{ for } x^{m-1},$$

and integrating each term separately.

Ex. 4. $\left(\int_{x=\infty} - \int_{x=0}\right) e^{-qx} \cos rx = \dfrac{q}{q^2 + r^2}$;

integrate with respect to r,

$$\therefore\ \left(\int_{x=\infty} - \int_{x=0}\right) \frac{e^{-qx}\sin rx}{x} = \tan^{-1}\left(\frac{r}{q}\right).$$

Hence if $q = 0$, $\left(\int_{x=\infty} - \int_{x=0}\right) \dfrac{\sin rx}{x} = \dfrac{\pi}{2}$, $\left(\int_{x=\infty} - \int_{x=0}\right) \cos rx = 0$.

Integrals such as these, and $\left(\int_{x=\infty} - \int_{x=0}\right) \sin rx = \dfrac{1}{r}$, (p. 115), that is of periodic quantities, have determined values, only when they are regarded as the limits of other integrals, of which some of the elements have vanished.

112. Secondly, we may apply the principle of differentiation under the sign of definite integration to discover new results.

Ex. 1. To find the value of $\left(\int_{x=\infty} - \int_{x=0}\right) e^{-a^2x^2}\cos rx$.

Let it be denoted by $\phi(r)$,

$$\text{then } d_r\phi(r) = \left(\int_{x=\infty} - \int_{x=0}\right)\left(-xe^{-a^2x^2}\sin rx\right).$$

But, integrating by parts,

$$\int_x\left(-xe^{-a^2x^2}\sin rx\right) = \frac{e^{-a^2x^2}}{2a^2}\sin rx - \frac{r}{2a^2}\int_x e^{-a^2x^2}\cos rx;$$

therefore, taking these integrals between the limits $x=0$, $x=\infty$, since the integrated part vanishes at each limit,

$$d_r\phi(r) = -\frac{r}{2a^2}\phi(r), \text{ or } d_r\{\log_\epsilon\phi(r)\} = -\frac{r}{2a^2};$$

$$\therefore\ \log_\epsilon\phi(r) - \log_\epsilon C = -\frac{r^2}{4a^2}, \text{ or } \phi(r) = Ce^{-\frac{r^2}{4a^2}}.$$

To determine C, let $r = 0$, $\therefore\ \phi(0) = \left(\int_{x=\infty} - \int_{x=0}\right) e^{-a^2x^2} = \dfrac{\sqrt{\pi}}{2a}$;

$$\therefore\ \left(\int_{x=\infty} - \int_{x=0}\right) e^{-a^2x^2}\cos rx = \frac{\sqrt{\pi}e^{-\frac{r^2}{4a^2}}}{2a}.$$

This result may be verified by substituting for $\cos rx$ its development, and integrating each term separately, by Art. 103.

If $a = 0$, we have as before $(\int_{x=\infty} - \int_{x=0}) \cos rx = 0$.

Ex. 2. To determine $\phi(c) = (\int_{x=\infty} - \int_{x=0}) e^{-x^2-\frac{c^2}{x^2}}$;

$$d_c\phi(c) = (\int_{x=\infty} - \int_{x=0}) \left(\frac{-2c}{x^2} e^{-x^2-\frac{c^2}{x^2}}\right)$$

$$= -2(\int_{z=\infty} - \int_{z=0}) e^{-z^2-\frac{c^2}{z^2}}, \text{ making } x = \frac{c}{z},$$

$$= -2\phi(c);$$

$\therefore\ \phi(c) = Ce^{-2c}$; and $\phi(0) = C = (\int_{x=\infty} - \int_{x=0}) e^{-x^2} = \frac{1}{2}\sqrt{\pi}$,

$$\therefore\ \phi(c) = \tfrac{1}{2}\sqrt{\pi}\,e^{-2c}.$$

Ex. 3. To determine $\phi(c) = (\int_{x=\infty} - \int_{x=0}) \frac{\cos cx}{1+x^2}$;

$$d_c^2\phi(c) = (\int_{x=\infty} - \int_{x=0}) \frac{-x^2\cos cx}{1+x^2} = (\int_{x=\infty} - \int_{x=0}) \frac{1-(1+x^2)}{1+x^2}\cos cx$$

$$= \phi(c) - (\int_{x=\infty} - \int_{x=0}) \cos cx = \phi(c);$$

$\therefore\ \{d_c\phi(c)\}^2 = \{\phi(c)\}^2 + C$, multiplying by $2d_c\phi(c)$, and integrating. To determine C, we must make $c = 0$.

But $\phi(0) = (\int_{x=\infty} - \int_{x=0}) \frac{1}{1+x^2} = \frac{1}{2}\pi$;

and $d_c\phi(c) = (\int_{x=\infty} - \int_{x=0}) \frac{-x\sin cx}{1+x^2} = (\int_{z=\infty} - \int_{z=0}) \frac{-z\sin z}{c^2+z^2}$, if $cx = z$;

$$\therefore\ d_{c=0}\phi(c) = (\int_{z=\infty} - \int_{z=0}) \frac{-\sin z}{z} = -\frac{\pi}{2};\ \therefore\ C = 0.$$

Hence $d_c\phi(c) = -\phi(c)$, since $d_c\phi(c)$ is always negative;

$\therefore\ \phi(c) = C'e^{-c} = \frac{\pi}{2}e^{-c}$, or $(\int_{x=\infty} - \int_{x=0}) \frac{\cos cx}{1+x^2} = \frac{1}{2}\pi e^{-c}$.

Hence, differentiating and integrating with respect to c,

$$\left(\int_{x=\infty} - \int_{x=0}\right) \frac{x \sin c x}{1 + x^2} = \tfrac{1}{2} \pi e^{-c},$$

$$\left(\int_{x=\infty} - \int_{x=0}\right) \frac{1}{x} \cdot \frac{\sin c x}{1 + x^2} = \frac{\pi}{2}(1 - e^{-c}).$$

113. We shall now give certain formulæ for approximating to the values of integrals by infinite series, which was the second object of this section. The fundamental formula is Bernouilli's series, which may be either obtained by integrating by parts, or deduced from Taylor's series in the following manner.

We have $f(x + h) = f(x) + \frac{h}{1} d_x f(x) + \frac{h^2}{1 \,.\, 2} d_x^2 f(x) + \&c.$

Let $f(x) = \int_x u$, $\therefore\ f(x + h) = \int_{x=x+h} u$,

$\int_{x=x+h} u$ denoting, as usual, the value of $\int_x u$ when in it $x + h$ is written for x; also $d_x f(x) = u$, $d_x^2 f(x) = d_x u$, &c.;

$$\therefore\ \int_{x=x+h} u = \int_x u + h u + \frac{h^2}{1 \,.\, 2} d_x u + \frac{h^3}{1 \,.\, 2 \,.\, 3} d_x^2 u + \&c. \quad (1).$$

Now let $h = -x$, $\therefore\ \int_{x=x+h} u = \int_{x=0} u = C$ a constant;

hence, substituting and transposing,

$$\int_x u = C + x u - \frac{x^2}{1 \,.\, 2} d_x u + \frac{x^3}{1 \,.\, 2 \,.\, 3} d_x^2 u - \&c.$$

If we apply this to the case $u = x^m$,

we have $d_x u = m x^{m-1}$, $d_x^2 u = m(m - 1) x^{m-2}$, &c.

$$\therefore\ \int_x x^m = C + x^{m+1} - \frac{m}{1 \,.\, 2} x^{m+1} + \frac{m(m - 1)}{1 \,.\, 2 \,.\, 3} x^{m+1} - \&c.$$

$$= C + x^{m+1}\left\{1 - \frac{m}{1 \,.\, 2} + \frac{m(m - 1)}{1 \,.\, 2 \,.\, 3} - \&c.\right\};$$

$$\text{but } (1-1)^{m+1} = 1 - (m+1) + \frac{(m+1)m}{1 \, . \, 2} - \&c. = 0,$$

$$\therefore \; \frac{1}{m+1} = 1 - \frac{m}{1 \, . \, 2} + \frac{m(m-1)}{1 \, . \, 2 \, . \, 3} - \&c.,$$

$$\therefore \; \int_x x^m = C + \frac{x^{m+1}}{m+1}.$$

114. Let u be a function of x that does not become infinite between the limits $x=a$, $x=a+b$, and in equation (1) of the preceding Art. let $h=b$, $x=a$,

$$\therefore \; \left(\int_{x=a+b} - \int_{x=a}\right) u = \frac{b}{1} u_{x=a} + \frac{b^2}{1 \, . \, 2} d_{x=a} u + \frac{b^3}{1 \, . \, 2 \, . \, 3} d^2_{x=a} u + \&c.$$

This is the value of the definite integral of u between the limits $x=a$, $x=a+b$, but will not be sufficiently accurate, unless b be very small.

If we regard u as the ordinate of a curve, $\int_{x=a+b} u - \int_{x=a} u$ represents its area between the ordinates corresponding to $x=a$ $x=a+b$; and $m+1$ terms of the above series expresses the area (terminated by the same ordinates) of the parabola, which has a contact of the m^{th} order with the curve at the extremity of the first ordinate; for the equation to such a parabola (taking the foot of the first ordinate as origin and h as abscissa) is

$$y = u_{x=a} + \frac{h}{1} d_{x=a} u + \frac{h^2}{1 \, . \, 2} d^2_{x=a} u + \&c. + \frac{h^m}{1 \, . \, 2 \, . \, 3 \, . \, m} d^m_{x=a} u,$$

since, as we perceive, the value of y and its m differential coefficients corresponding to $h=0$, that is $x=a$, would be respectively equal to u and its m differential coefficients corresponding to $x=a$; and this, integrated with respect to h between the limits $h=0$, $h=b$, gives the above series.

115. To obtain greater accuracy, let the difference between the limits be divided into n equal intervals, that is, let the limits be a, and $b = a + nh$; hence in equation (1) Art. 113. changing x into a, $a + h$, $a + 2h$, &c., successively,

$$\left(\int_{x=a+h} - \int_{x=a}\right) u = hu_{x=a} + \frac{h^2}{1\,.\,2}\, d_{x=a}u + \frac{h^3}{1\,.\,2\,.\,3}\, d^2_{x=a}u + \&\text{c.},$$

$$\left(\int_{x=a+2h} - \int_{x=a+h}\right) u = hu_{x=a+h} + \frac{h^2}{1.2}\, d_{x=a+h}u + \frac{h^3}{1.2.3}\, d^2_{x=a+h}u + \&\text{c.}$$

$$\&\text{c.} = \&\text{c.}$$

$$\left(\int_{x=b} - \int_{x=b-h}\right) u = hu_{x=b-h} + \frac{h^2}{1.2}\, d_{x=b-h}u + \frac{h^3}{1\,.\,2\,.\,3}\, d^2_{x=b-h}u + \&\text{c.},$$

$$\therefore \text{ by addition, } \left(\int_{x=b} - \int_{x=a}\right) u = h\,(u_{x=a} + u_{x=a+h} + \&\text{c.} + u_{x=b-h})$$

$$+ \frac{h^2}{1\,.\,2}\,(d_{x=a}u + d_{x=a+h}u + \&\text{c.} + d_{x=b-h}u)$$

$$+ \frac{h^3}{1.2.3}\,(d^2_{x=a}u + d^2_{x=a+h}u + \&\text{c.} + d^2_{x=b-h}u) + \&\text{c., where } h = \frac{b-a}{n}.$$

This series may be made convergent by taking h sufficiently small.

116. In the preceding Art. we may write the first line

$$\left(\int_{x=a+h} - \int_{x=a}\right) u = hu_{x=a} + h^{1+r}R_0,$$

denoting by r (which usually will $= 1$) a positive quantity, in order to embrace every case, and by R_0 a function of a and h; then the result may be written

$$\left(\int_{x=b} - \int_{x=a}\right) u = h\,(u_{x=a} + u_{x=a+h} + \&\text{c.} + u_{x=b-h})$$

$$+ h^{1+r}(R_0 + R_1 + \&\text{c.} + R_{n-1}).$$

Let M be the greatest value (not considering the sign) which can be assumed by any one of the quantities R_0, R_1, &c. not one of which can become infinite because u is finite from $x = a$ to $x = a + nh$;

$$\therefore\ \left(\int_{x=b} - \int_{x=a}\right) u - h\,(u_{x=a} + u_{x=a+h} + \text{\&c.} + u_{x=b-h})$$
$$< h^{1+r} n M < h^r (b-a) M.$$

But when n is infinite, $h = 0$,

$$\therefore\ \left(\int_{x=b} - \int_{x=a}\right) u = \text{limit of } (h u_{x=a} + h u_{x=a+h} + \text{\&c.} + h u_{x=b-h}).$$

117. Again, making h negative in equation (1) Art. 113, and changing x into b, $b - h$, &c., successively,

$$\left(\int_{x=b} - \int_{x=b-h}\right) u = h u_{x=b} - \frac{h^2}{1.2} d_{x=b} u + \frac{h^3}{1.2.3} d^2_{x=b} u - \text{\&c.}$$

$$\left(\int_{x=b-h} - \int_{x=b-2h}\right) u = h u_{x=b-h} - \frac{h^2}{1.2} d_{x=b-h} u + \frac{h^3}{1.2.3} d^2_{x=b-h} u - \text{\&c.}$$

&c. = &c.

$$\left(\int_{x=a+h} - \int_{x=a}\right) u = h u_{x=a+h} - \frac{h^2}{1.2} d_{x=a+h} u + \frac{h^3}{1.2.3} d^2_{x=a+h} u - \text{\&c.}$$

$$\therefore\ \left(\int_{x=b} - \int_{x=a}\right) u = h\,(u_{x=b} + u_{x=b-h} + \text{\&c.} + u_{x=a+h})$$
$$- \frac{h^2}{1.2} (d_{x=b} u + d_{x=b-h} u + \text{\&c.} + d_{x=a+h} u)$$
$$+ \frac{h^3}{1.2.3} (d^2_{x=b} u + d^2_{x=b-h} u + \text{\&c.} + d^2_{x=a+h} u) - \text{\&c.}$$

118. If we suppose all the quantities $u_{x=a}$, $d_{x=a} u$, &c. $u_{x=a+h}$, $d_{x=a+h} u$, &c. to remain positive from $x = a$ to $x = b$, the above series gives a value too large when we stop at a term preceded by a positive sign, and the former series gives a value too small; consequently by adding these values of $\left(\int_{x=b} - \int_{x=a}\right) u$ together, and taking half the sum, we shall obtain a still nearer approximation; this gives

$$\left(\int_{x=b} - \int_{x=a}\right) u = h\left\{\tfrac{1}{2}(u_{x=a} + u_{x=b}) + u_{x=a+h} + \text{\&c.} + u_{x=b-h}\right\}$$
$$+ \frac{h^2}{1.2}\,\tfrac{1}{2}\,(d_{x=a} u - d_{x=b} u)$$

$$+\frac{h^3}{1.2.3}\left\{\tfrac{1}{2}(d^2_{x=a}u + d^2_{x=b}u) + d^2_{x=a+h}u + \&c. + d^2_{x=b-h}u\right\}$$

$$+\frac{h^4}{1.2.3.4}\tfrac{1}{2}(d^3_{x=a}u - d^3_{x=b}u) + \&c.$$

119. If we regard the proposed function u as the ordinate of a curve, $u_{x=a}$, $u_{x=a+h}$, &c., $u_{x=b}$ will represent $(n+1)$ equidistant ordinates of the curve, and $(\int_{x=b} - \int_{x=a})u$ the area contained between the extreme ordinates; and by the first series this area will be given in terms of the first, second, n^{th} ordinates and their differential coefficients, and by the second, in terms of the second, third, $(n+1)^{\text{th}}$ ordinates and their differential coefficients; hence the first terms of these series are respectively the sums of the inscribed, and circumscribed parallelograms; and the first term of the third series is the sum of the inscribed trapezia. Also by Art. 114. it appears that the first and second series are the expressions for the areas of the polygon formed by the arcs of the osculating parabolas comprised between each pair of consecutive ordinates.

120. There are several expansions which may be readily obtained from integration by series.

Ex. 1. $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \&c.,$

$$\therefore \log_\epsilon(1+x) = \int_x \frac{1}{1+x} = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \&c.;$$

but when $x = 0$, $\log_\epsilon(1+x) = \log_\epsilon 1 = 0$, $\therefore C = 0$;

$$\therefore \log_\epsilon(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \&c.$$

Ex. 2. $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \&c.,$

$$\therefore \tan^{-1}x = \int_x \frac{1}{1+x^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \&c.; \quad (C = 0).$$

Again, $$\frac{1}{1+x^2} = \frac{\dfrac{1}{x^2}}{1+\dfrac{1}{x^2}} = \frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{x^6} - \&c.$$

$$\therefore\ \tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \&c.,$$

determining the constant by making $x = \infty$; the latter series is an approximation to the value of $\tan^{-1} x$ when x is very large.

Ex. 3. $$\frac{1}{\sqrt{1-x^2}} = 1 + \tfrac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \&c.,$$

$$\therefore\ \sin^{-1} x = \int_x \frac{1}{\sqrt{1-x^2}}$$

$$= x + \tfrac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \&c.\ (C = 0).$$

121. When the expression $(a+bx^n)^{\frac{p}{q}}x^{m-1}$ is integrable only by development, expand $\left(1+\dfrac{bx^n}{a}\right)^{\frac{p}{q}}$ by the binomial theorem, and integrate each term; the result is

$$\int_x (a+bx^n)^{\frac{p}{q}}x^{m-1}$$

$$= a^{\frac{p}{q}}\left\{\frac{x^m}{m} + \frac{pb}{qa}\frac{x^{m+n}}{m+n} + \frac{p(p-q)b^2}{1.2q^2a^2}\frac{x^{m+2n}}{m+2n} + \&c.\right\};$$

this being an ascending series, enables us to approximate to the value of the integral when x is small. If a descending series be required, write the expression $(b+ax^{-n})^{\frac{p}{q}}x^{m+\frac{np}{q}-1}$, expand, and integrate as before.

122. To integrate $\dfrac{1}{\sqrt{1-c^2(\sin x)^2}}$, the origin of the integral being $x = 0$, and $c < 1$.

Expanding by the binomial theorem, we find

$$\frac{1}{\sqrt{1-c^2(\sin x)^2}}=1+\tfrac{1}{2}c^2(\sin x)^2+\frac{1\,.\,3}{2\,.\,4}c^4(\sin x)^4+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}c^6(\sin x)^6+\&\text{c.}$$

But (Art. 75) we have the formula of reduction

$$\textstyle\int_x (\sin x)^n = -\frac{1}{n}\cos x\,(\sin x)^{n-1} + \frac{n-1}{n}\int_x (\sin x)^{n-2};$$

$$\therefore\ \textstyle\int_x (\sin x)^2 = -\tfrac{1}{2}\cos x\sin x + \frac{x}{2},$$

$$\textstyle\int_x (\sin x)^4 = -\tfrac{1}{4}\cos x\,(\sin x)^3 + \tfrac{3}{4}\int_x (\sin x)^2$$

$$= -\tfrac{1}{4}\cos x\,(\sin x)^3 - \frac{1\,.\,3}{2\,.\,4}\cos x\sin x + \frac{1\,.\,3}{2\,.\,4}x;$$

&c. = &c.

$$\therefore\ \int_x \frac{1}{\sqrt{1-c^2(\sin x)^2}} = x + \frac{c^2}{2}\left(-\tfrac{1}{2}\cos x\sin x + \frac{x}{2}\right)$$

$$+\frac{1\,.\,3}{2\,.\,4}c^4\left\{-\tfrac{1}{4}\cos x(\sin x)^3 - \frac{1\,.\,3}{2\,.\,4}\cos x\sin x + \frac{1\,.\,3}{2\,.\,4}x\right\} + \&\text{c.}$$

If the value of the definite integral between the limits $x=0$, $x=\frac{\pi}{2}$, be required,

$$\text{since } \left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)(\sin x)^{2n} = \frac{(2n-1)(2n-3)\ldots 1}{2n(2n-2)\ldots 2}\,\frac{\pi}{2};$$

$$\therefore\ \left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)\frac{1}{\sqrt{1-c^2(\sin x)^2}} = \frac{\pi}{2} + \frac{1}{2}c^2\,.\,\frac{1}{2}\frac{\pi}{2}$$

$$+\frac{1\,.\,3}{2\,.\,4}c^4\,.\,\frac{1\,.\,3}{2\,.\,4}\frac{\pi}{2} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}c^6\,.\,\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{\pi}{2} + \&\text{c.}$$

$$= \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2 c^2 + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2 c^4 + \left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2 c^6 + \&\text{c.}\right\}.$$

123. To integrate $\sqrt{1-c^2(\sin x)^2}$, c being < 1.

Proceeding as above, $\sqrt{1-c^2(\sin x)^2} = 1 - \frac{1}{2}c^2(\sin x)^2$

$$-\frac{1.1}{2.4}c^4(\sin x)^4 - \frac{1.1.3}{2.4.6}c^6(\sin x)^6 - \&c.,$$

$$\therefore \int_x \sqrt{1-c^2(\sin x)^2} = x - \frac{c^2}{2}\left(-\frac{1}{2}\cos x \sin x + \frac{x}{2}\right)$$

$$-\frac{1.1}{2.4}c^4\left\{-\frac{1}{4}\cos x(\sin x)^3 - \frac{1.3}{2.4}\cos x \sin x + \frac{1.3}{2.4}x\right\} - \&c.;$$

$$\text{and } \left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)\sqrt{1-c^2(\sin x)^2}$$

$$= \frac{\pi}{2}\left\{1 - \left(\frac{1}{2}\right)^2\frac{c^2}{1} - \left(\frac{1.3}{2.4}\right)^2\frac{c^4}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2\frac{c^6}{5} - \&c.\right\}.$$

0 124. These series always converge since $c<1$; but the convergence is slow if c be considerable, and a more converging series may be found thus.

$$\text{Let } n = \frac{1-\sqrt{1-c^2}}{1+\sqrt{1-c^2}}, \quad \therefore \frac{1-n}{1+n} = \sqrt{1-c^2},$$

$$c^2 = 1 - \left(\frac{1-n}{1+n}\right)^2 = \frac{4n}{(1+n)^2};$$

$$\therefore \sqrt{1-c^2(\sin x)^2} = \sqrt{1 - \frac{4n}{(1+n)^2}(\sin x)^2}$$

$$= \frac{1}{1+n}\sqrt{(1+n)^2 - 2n(1-\cos 2x)} = \frac{1}{1+n}\sqrt{1+2n\cos 2x + n^2}$$

$$= \frac{1}{1+n}\sqrt{1+n(z+z^{-1})+n^2} \text{ (making } 2\cos 2x = z + z^{-1})$$

$$= \frac{1}{1+n}\sqrt{1+nz}\,.\,\sqrt{1+nz^{-1}}$$

$$= \frac{1}{1+n}\left\{1 + \frac{1}{2}nz - \frac{1.1}{2.4}n^2z^2 + \frac{1.1.3}{2.4.6}n^3z^3 - \&c.\right\}$$

$$\times\left\{1 + \frac{1}{2}nz^{-1} - \frac{1.1}{2.4}n^2z^{-2} + \frac{1.1.3}{2.4.6}n^3z^{-3} - \&c.\right\}$$

$$= \frac{1}{1+n}\left\{1 + \left(\frac{1}{2}\right)^2 n^2 + \left(\frac{1.1}{2.4}\right)^2 n^4 + \left(\frac{1.1.3}{2.4.6}\right)^2 n^6 + \&c.\right.$$

$$\left.+ A(z+z^{-1}) + B(z^2+z^{-2}) + \&c.\right\}$$

$$= \frac{1}{1+n}\left\{1 + \left(\frac{1}{2}\right)^2 n^2 + \left(\frac{1.1}{2.4}\right)^2 n^4 + \left(\frac{1.1.3}{2.4.6}\right)^2 n^6 + \&c.\right.$$

$$\left.+ 2A\cos 2x + 2B\cos 4x + \&c.\right\};$$

$$\therefore\ \left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right)\sqrt{1-c^2(\sin x)^2}$$

$$= \frac{\pi}{2(1+n)}\left\{1 + \left(\frac{1}{2}\right)^2 n^2 + \left(\frac{1.1}{2.4}\right)^2 n^4 + \left(\frac{1.1.3}{2.4.6}\right)^2 n^6 + \&c.\right\};$$

which converges more rapidly than the former series, because $\frac{n}{c} = \frac{c}{(1+\sqrt{1-c^2})^2}$ is less than 1, or $n < c$.

125. Similarly,

$$\frac{1}{\sqrt{1-c^2(\sin x)^2}} = (1+n)(1+nz)^{-\frac{1}{2}}(1+nz^{-1})^{-\frac{1}{2}}$$

$$= (1+n)\left\{1 - \frac{1}{2}nz + \frac{1.3}{2.4}n^2z^2 - \frac{1.3.5}{2.4.6}n^3z^3 + \&c.\right\}$$

$$\times\left\{1 - \frac{1}{2}nz^{-1} + \frac{1.3}{2.4}n^2z^{-2} - \frac{1.3.5}{2.4.6}n^3z^{-3} + \&c.\right\}$$

$$= (1+n)\left\{1 + \left(\frac{1}{2}\right)^2 n^2 + \left(\frac{1.3}{2.4}\right)^2 n^4 + \left(\frac{1.3.5}{2.4.6}\right)^2 n^6 + \&c.\right.$$

$$\left.+ 2A\cos 2x + 2B\cos 4x + \&c.\right\};$$

$$\therefore \left(\int_{x=\frac{\pi}{2}} - \int_{x=0}\right) \frac{1}{\sqrt{1 - c^2 (\sin x)^2}}$$

$$= \frac{\pi (1+n)}{2} \left\{1 + \left(\frac{1}{2}\right)^2 n^2 + \left(\frac{1 \,.\, 3}{2 \,.\, 4}\right)^2 n^4 + \left(\frac{1 \,.\, 3 \,.\, 5}{2 \,.\, 4 \,.\, 6}\right)^2 n^6 + \&c.\right\}.$$

126. The integrals treated of in the four preceding Articles are very important, being the elementary integrals of elliptic functions. On these depends the rectification of the ellipse and hyperbola as we shall shew; besides there are many integrals that occur in philosophical problems which may be conveniently reduced to them.

Ex. 1. $\displaystyle\int_x \frac{1}{\sqrt{1 - x^4}} = -\frac{1}{\sqrt{2}} \int_z \frac{1}{\sqrt{1 - \frac{1}{2} (\sin z)^2}}$, making $x = \cos z$;

$$\therefore \left(\int_{x=1} - \int_{x=0}\right) \frac{1}{\sqrt{1 - x^4}} = \frac{1}{\sqrt{2}} \left(\int_{z=\frac{\pi}{2}} - \int_{z=0}\right) \frac{1}{\sqrt{1 - \frac{1}{2} (\sin z)^2}}$$

$$= \frac{\pi}{2\sqrt{2}} \left\{1 + (\tfrac{1}{2})^2 \cdot \tfrac{1}{2} + \left(\frac{1 \,.\, 3}{2 \,.\, 4}\right)^2 \cdot \frac{1}{2^2} + \&c.\right\}.$$

Ex. 2. $\displaystyle\int_x \frac{1}{\sqrt{1 + x^4}} = \frac{1}{2} \int_z \frac{1}{\sqrt{1 - \frac{1}{2} (\sin z)^2}}$, where $x = \tan \dfrac{z}{2}$.

Ex. 3. $\displaystyle u = \int_x \frac{1}{\sqrt{(2ax - x^2)(b - x)}} = \int_z \frac{1}{\sqrt{b - a \operatorname{versin} z}}$,

making $\dfrac{x}{a} = \operatorname{versin} z$;

if, therefore, $\dfrac{2a}{b} < 1$, $\displaystyle u = \frac{1}{\sqrt{b}} \int_z \frac{1}{\sqrt{1 - \frac{2a}{b} \left(\sin \frac{z}{2}\right)^2}}$.

But if $\dfrac{2a}{b} > 1$, let $\dfrac{b}{2a} = c^2$, and $\sin \dfrac{z}{2} = c \sin \phi$,

$$\therefore u = \frac{2c}{\sqrt{b}} \int_\phi \frac{1}{\sqrt{1 - c^2 (\sin \phi)^2}}.$$

Ex. 4. $\int_x \frac{1}{\sqrt{(a-x)(x-b)(x+c)}} = \frac{-2}{\sqrt{a+c}} \int_z \frac{1}{\sqrt{1 - \frac{a-b}{a+c} (\sin z)^2}}$,

making $\sin z = \sqrt{\frac{a-x}{a-b}}$; it being supposed that $a > b$, and therefore a is the greatest, and b the least of the admissible values of x.

127. To integrate $\log_\epsilon (1 + n \cos x)$.

Let $1 + n\cos x = \alpha^2 + \beta^2 + 2\alpha\beta \cos x = \alpha^2 (1 + mz)(1 + mz^{-1})$,

$$\text{making } m = \frac{\beta}{\alpha},\ 2\cos x = z + z^{-1};$$

$$\therefore \log_\epsilon (1 + n\cos x) = \log_\epsilon \alpha^2 + (mz - \tfrac{1}{2} m^2 z^2 + \tfrac{1}{3} m^3 z^3 - \&c.)$$

$$+ (mz^{-1} - \tfrac{1}{2} m^2 z^{-2} + \tfrac{1}{3} m^3 z^{-3} - \&c.)$$

$$= \log_\epsilon \alpha^2 + 2m \cos x - \frac{m^2}{2} 2\cos 2x + \frac{m^3}{3} 2\cos 3x - \&c. \quad (1)$$

$$\therefore \int_x \log_\epsilon (1 + n\cos x) = x \log_\epsilon \alpha^2 + 2m \sin x$$

$$- \frac{m^2}{2} \sin 2x + \frac{m^3}{3} \frac{2}{3} \sin 3x - \&c.;$$

where (since $\alpha^2 + \beta^2 = 1$, $2\alpha\beta = n$),

$$m = \frac{\beta}{\alpha} = \frac{\sqrt{1+n} - \sqrt{1-n}}{\sqrt{1+n} + \sqrt{1-n}} = \frac{1 - \sqrt{1-n^2}}{n},$$

$$\text{and } \alpha^2 = \tfrac{1}{4} (\sqrt{1+n} + \sqrt{1-n})^2$$

$$= \frac{1 + \sqrt{1-n^2}}{2} = \frac{n^2}{2(1 - \sqrt{1-n^2})} = \frac{n}{2m}.$$

128. The above result leads to the values of several remarkable Definite Integrals.

First, $(\int_{x=\pi} - \int_{x=0}) \log_\epsilon (1 + n \cos x) = \pi \log_\epsilon \left(\frac{1 + \sqrt{1-n^2}}{2}\right)$.

Also since

$$\int_x \log_\epsilon (1 + n \cos x) = x \log_\epsilon (1 + n \cos x) + n \int_x \frac{x \sin x}{1 + n \cos x},$$

$$\therefore \left(\int_{x=\pi} - \int_{x=0}\right) \frac{x \sin x}{1 + n \cos x} = \frac{\pi}{n} \log_\epsilon \left(\frac{1 + \sqrt{1 - n^2}}{2(1 - n)}\right).$$

Again, differentiating equation (1), and replacing x by cx,

$$\frac{n \sin cx}{1 + n \cos cx} = 2 (m \sin cx - m^2 \sin 2cx + m^3 \sin 3cx - \&\text{c.}),$$

therefore (Ex. 3. Art. 112),

$$\left(\int_{x=\infty} - \int_{x=0}\right) \frac{x \sin cx}{(1+x^2)(1+n \cos cx)} = \frac{\pi}{n} (m e^{-c} - m^2 e^{-2c} + m^3 e^{-3c} - \&\text{c.})$$

$$= \frac{m\pi}{n(e^c + m)} = \frac{\pi}{n + e^c (1 + \sqrt{1 - n^2})}.$$

129. To integrate $(1 + e \cos x)^n$.

This expression, which is another form of

$$(a^2 + 2aa' \cos x + a'^2)^n,$$

often occurs in the applications of the Integral Calculus; when n is a fraction, in order to be integrated, it must be developed in a series.

Let $(1 + e \cos x)^n = \frac{1}{2} a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \&\text{c.}$, therefore, taking the differential coefficient of the logarithm of each side,

$$\frac{ne \sin x}{1 + e \cos x} = \frac{a_1 \sin x + \&\text{c.} + (r-1) a_{r-1} \sin (r-1)x + r a_r \sin rx + \&\text{c.}}{\frac{1}{2} a_0 + a_1 \cos x + \&\text{c.} + a_{r-1} \cos (r-1)x + a_r \cos rx + \&\text{c.}},$$

Multiply crosswise, and retain only the products which when resolved will contain $\sin rx$,

$$\therefore\ ne \sin x \left\{a_{r-1} \cos (r-1)x + a_{r+1} \cos (r+1)x\right\} + \&\text{c.} = r a_r \sin rx$$
$$+ e \cos x \left\{(r-1) a_{r-1} \sin (r-1)x + (r+1) a_{r+1} \sin (r+1)x\right\} + \&\text{c.};$$

therefore, equating coefficients of $\sin rx$,

$$\frac{ne}{2}(a_{r-1} - a_{r+1}) = ra_r + \frac{e}{2}\{(r-1)\,a_{r-1} + (r+1)\,a_{r+1}\};$$

$$\therefore\ a_{r+1} = \frac{a_{r-1}(n-r+1)\,e - 2ra_r}{(n+r+1)\,e}$$

which holds when $r = 1$, as may be easily shewn.

Hence, a_2, a_3, a_4, &c. can be deduced from the preceding coefficients.

When n is a positive integer, all the coefficients after a_n vanish; for $\cos nx$ involves $(\cos x)^n$, and this is the highest power of $\cos x$ that can enter.

It only remains therefore to determine the two first coefficients $\frac{1}{2}a_0$, and a_1. But $(1 + e\cos x)^n = 1 + ne\cos x$

$$+\frac{n(n-1)}{1.2}e^2(\cos x)^2 + \frac{n(n-1)(n-2)}{1.2.3}e^3(\cos x)^3 + \&c.$$

$$= 1 + ne\cos x + \frac{n(n-1)}{1.2}\frac{e^2}{2}(1 + \cos 2x)$$

$$+\frac{n(n-1)(n-2)}{1.2.3}\frac{e^3}{4}(\cos 3x + 3\cos x)$$

$$+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\frac{e^4}{8}(\cos 4x + 4\cos 2x + 3) + \&c.,$$

$$\therefore\ \tfrac{1}{2}a_0 = 1 + \frac{n(n-1)}{1.2}\frac{e^2}{2} + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\frac{3e^4}{8} + \&c.,$$

$$a_1 = e\left\{n + \frac{n(n-1)(n-2)}{1.2.3}\frac{3e^2}{4} + \&c.\right\}.$$

Of course these series for $\frac{1}{2}a_0$, a_1, terminate, when n is a positive integer.

Hence, all the coefficients being known, we have

$$\int_x (1 + e\cos x)^n = \tfrac{1}{2}a_0 x + a_1 \sin x + \tfrac{1}{2}a_2 \sin 2x + \&c.$$

130. It may be observed that

$$a_r = \frac{2}{\pi}\left({}_{x=\pi} - \int_{x=0}\right)(1 + e\cos x)^n \cos rx.$$

For the general term of

$$\int_x (1 + e\cos x)^n \cos rx \text{ would be } a_m \int_x \cos mx \cos rx,$$

$$\text{which } = \tfrac{1}{2}a_m \left\{\frac{\sin(m+r)x}{m+r} + \frac{\sin(m-r)x}{m-r}\right\} + C;$$

and this would vanish between the limits $x = 0$, $x = \pi$, in every case except when $m = r$, when it would $= \frac{a_r}{2}\pi$,

for the fraction $\frac{\sin(m-r)\pi}{m-r} = \frac{(m-r)\pi}{m-r} = \pi$, when $m = r$.

131. From the coefficients of the expansion of $(1+e\cos x)^{-n}$, those of the expansion of $(1+e\cos x)^{-n-1}$ are immediately found by differentiation.

$$\text{For } (1 + e\cos x)^{-n-1} = (1+e\cos x)^{-n} - e\cos x\,(1 + e\cos x)^{-n-1}$$

$$= (1 + e\cos x)^{-n} + \frac{e}{n}d_e(1 + e\cos x)^{-n} = (a_0 + \frac{e}{n}d_e a_0)$$

$$+ (a_1 + \frac{e}{n}d_e a_1)\cos x + (a_2 + \frac{e}{n}d_e a_2)\cos 2x + \&c.$$

Or, without differentiation, if

$$(1 + e\cos x)^{-n} = \tfrac{1}{2}a_0 + a_1\cos x + \&c. + a_r \cos rx + \&c.$$
$$(1 + e\cos x)^{-n-1} = \tfrac{1}{2}b_0 + b_1\cos x + \&c. + b_r \cos rx + \&c.$$

it may be easily proved that

$$n(1 - e^2)b_r = (n - r)a_r - (n + r - 1)ea_{r-1}.$$

132. Hence, using the notation of Art. ~~91~~ 127, since $\frac{1}{1+e\cos x}$

$$= \frac{1}{a^2}\frac{1}{(1+mz)(1+mz^{-1})} = \frac{1}{a^2(1-m^2)}\left\{\frac{1}{1+mz} - \frac{mz^{-1}}{1+mz^{-1}}\right\}$$

$$= \frac{1}{a^2-\beta^2}\{1 - 2m\cos x + 2m^2\cos 2x - 2m^3\cos 3x + \&c.\}$$

$$= \frac{1}{\sqrt{1-e^2}}\left\{1 - \frac{2(1-\sqrt{1-e^2})}{e}\cos x + \frac{2(1-\sqrt{1-e^2})^2}{e^2}\cos 2x - \&c.\right\},$$

we can, by successive differentiations, determine the coefficients of $(1+e\cos x)^{-2}$, &c.

133. To find $\int_t e^{-t^2}$ in a series, between the limits $t=0$, $t=t_1$.

Integrating by parts,

$$\int_t e^{-t^2} = \int_t \frac{-1}{2t}\, d_t(e^{-t^2}) = -\frac{1}{2t}e^{-t^2} - \tfrac{1}{2}\int_t e^{-t^2}t^{-2},$$

$$\int_t e^{-t^2}t^{-2} = -\tfrac{1}{2}e^{-t^2}t^{-3} - \tfrac{3}{2}\int_t e^{-t^2}t^{-4}, \text{ \&c.},$$

$$\therefore \int_t e^{-t^2} = -\frac{e^{-t^2}}{2t}\left(1 - \tfrac{1}{2}t^{-2} + \frac{1.3}{2^2}t^{-4} - \frac{1.3.5}{2^3}t^{-6} + \&c.\right) + C;$$

$$\therefore \int_{t=\infty} e^{-t^2} = 0 + C;$$

$$\therefore \left(\int_{t=t_1} - \int_{t=\infty}\right) e^{-t^2} = -\frac{e^{-t_1^2}}{2t_1}\left(1 - \tfrac{1}{2}t_1^{-2} + \frac{1.3}{2^2}t_1^{-4} - \&c.\right)$$

But $\left(\int_{t=\infty} - \int_{t=0}\right) e^{-t^2} = \tfrac{1}{2}\sqrt{\pi}$,

Therefore by addition,

$$\left(\int_{t=t_1} - \int_{t=0}\right) e^{-t^2} = \tfrac{1}{2}\sqrt{\pi} - \frac{e^{-t_1^2}}{2t_1}\left(1 - \tfrac{1}{2}t_1^{-2} + \frac{1.3}{2^2}t_1^{-4} - \&c.\right).$$

SECTION VIII.

AREAS AND LENGTHS OF CURVES, AND VOLUMES AND AREAS OF THE SURFACES OF FIGURES GENERATED BY THEIR REVOLUTION.

ART. 134. WE now come to one of the applications of the Integral Calculus which usually forms a portion of treatises on the subject, both on account of the interesting illustration it furnishes of many of the methods already explained, and of the utility of the results to which it leads.

135. If (s) represent the length of the arc of a plane curve, A the area contained by the arc, the ordinates at its extremities, and the intercepted portion of the axis of x, and V and S the volume and area of the curved surface of the figure generated by the revolution of the curve about the axis of x, it is proved in the Differential Calculus, that

$$d_x s = \sqrt{1+(d_x y)^2},\quad d_x A = y,\quad d_x V = \pi y^2,\quad d_x S = 2\pi y\sqrt{1+(d_x y)^2};$$

therefore, inverting the formulæ,

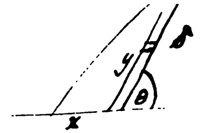
$$s = \textstyle\int_x \sqrt{1+(d_x y)^2},\quad A = \int_x y,\quad V = \pi\int_x y^2,\quad S = 2\pi\int_x y\sqrt{1+(d_x y)^2}.$$

When the curve is given, y and $d_x y$ are known functions of x, and therefore all the above integrals are of the form $\int_x u$ (u being a function of x); the integrals must of course be taken between the limits corresponding to the boundary of the figure whose area or volume is required.

If the area be bounded by another curve, whose ordinate is y_1, in place of the axis of x,

$$\text{then } A = \textstyle\int_x y - \int_x y_1 = \int_x (y - y_1),\quad V = \pi\int_x (y^2 - y_1^2).$$

dx, dy respectively
∴ area of d = dx . dy sin θ
∴ whole area = $\iint dx\, dy \sin\theta$
= $\int_x y . \sin\theta = \sin\theta \int_x y$



Also, if the axes of the co-ordinates, instead of being rectangular, be inclined to one another at an angle θ, then $d_x A = y \sin\theta$, and consequently $A = \sin\theta \int_x y$.*

136. There is another mode of expressing the length of a curve which deserves notice.

Let $CN = x$, $PN = y$, $AP = s$, $\angle\, NPT = NCY = \theta$,

CY (perpendicular to the tangent at P) $= p$, $PY = u$, (Fig. 3 and 4),

$$\therefore\ p = CP \cos(\theta \mp PCN) = x \cos\theta \pm y \sin\theta,$$

$$u = CP \sin(\theta \mp PCN) = x \sin\theta \mp y \cos\theta,$$

$$d_x y = \mp \cot\theta, \quad d_x s = \mp \operatorname{cosec}\theta,$$

$$\therefore\ d_\theta p = \cos\theta\, d_\theta x - x \sin\theta - \cot\theta\, d_\theta x \,.\, \sin\theta \pm y \cos\theta$$

$$= - x \sin\theta \pm y \cos\theta = - u,$$

$$d_\theta^2 p = - x \cos\theta - \sin\theta\, d_\theta x \mp y \sin\theta - \cot\theta\, d_\theta x \,.\, \cos\theta$$

$$= - p - \operatorname{cosec}\theta\, d_\theta x = - p \pm d_\theta s, \quad \therefore\ d_\theta p = - \textstyle\int_\theta p \pm s,$$

$$\text{or } \pm s = d_\theta p + \textstyle\int_\theta p,$$

+ or − according as the curve is concave or convex to the origin;

$$\text{also } u \pm s = \textstyle\int_\theta p.$$

0 = cos θ ± $\frac{dy}{dx}$
∴ $\frac{dy}{dx}$ = ∓ c

CP

If the curve cut the axis at right angles at A, no constant is necessary after integration.

This general property of curves is chiefly of use in finding the lengths of elliptic and hyperbolic arcs; it may be also used,

(1) To determine the length of any curve whose equation is given; for from that equation, together with $d_x y = - \cot\theta$, we can find x, y, and therefore $p = x \cos\theta + y \sin\theta$, in terms of θ; and then $s = d_\theta p + \int_\theta p$.

(2) To find a curve whose arc shall represent a proposed integral; for if it be put under the form $\int_\theta p$, where p is

* p = CP cos θ . cos PCN ± CP sin θ . sin PCN
but CP = $\frac{x}{\cos PCN}$ and = $\frac{y}{\sin PCN}$
∴ p = x cos θ ± y sin θ }
[illegible]

a function of θ, the equation to the required curve is found by eliminating θ between the equations

$$x = p\cos\theta - d_\theta p\sin\theta,\quad y = p\sin\theta + d_\theta p\cos\theta.$$

If p be such that $\int_\theta p$ is algebraic, we obtain a curve the length of whose arc is expressed by an algebraic quantity.

137. We shall now apply the above formulæ to curves of the second order, the areas and volumes of which are necessary to be known, and to certain others of the more common curves.

Ex. 1. To find the area of a circle.

We have already seen (Art. 97.), that the area of a quadrant $= \frac{1}{4}\pi a^2$, therefore the area of a circle whose radius $= a$, is πa^2.

Hence the area of a sector of a circle, whose angle is α, will $= \frac{1}{2}a^2\alpha$; for it will be the same part of the area of the circle, that its angle is of four right angles.

In addition to the expressions already found for $\int_x\sqrt{a^2-x^2}$, and $\int_x\sqrt{2ax-x^2}$ (Art. 25.), we may notice the following; $\int_x\sqrt{a^2-x^2}$ = circular area (radius $= a$, abscissa from center $= x$) $+ C$; and $\int_x\sqrt{2ax-x^2}$ = circular area (radius $= a$, abscissa from extremity of diameter $= x$) $+ C$.

Ex. 2. Let AB (Fig. 5.) be an elliptic quadrant whose semiaxes are $AC=a$, $BC=b$; AD the quadrant of a concentric circle whose diameter is the major axis, $AN = x$, $PN = y$;

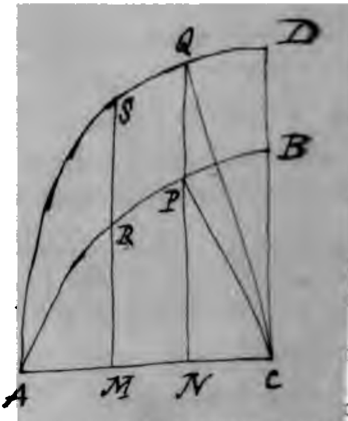
$$\text{therefore, area } ANP = \int_x \frac{b}{a}\sqrt{2ax-x^2} = \frac{b}{a}\int_x\sqrt{2ax-x^2}$$

$$= \frac{b}{a}\,(\text{circular area } ANQ);$$

$$\text{hence also area } AMR = \frac{b}{a}\,(\text{circular area } AMS),$$

$$\therefore \text{ area } MRPN = \frac{b}{a}\,(\text{circular area } SMNQ);$$

fig 6



between two parallel ordinates, bears corresponding portion of the circular $PNC = \frac{b}{a}$ (triangle QNC),

$CP = \frac{b}{a}$ (circular sector ACQ);

ertex of the sectors be at any other d of C.

e whole ellipse $= \frac{b}{a}\pi a^2 = \pi ab$, or is le whose radius $= \sqrt{ab}$, a mean proportional between the semi-axes. Since the whole area is divided into four equal sectors by any pair of conjugate diameters, if a', b', denote the lengths of two $\frac{1}{2}$ conjugate diameters inclined at an angle γ, the sector included by them

$$= \tfrac{1}{4}\pi ab = \tfrac{1}{4}\pi a'b' \sin\gamma.$$ conic 6

Ex. 3. To find the length of a circular arc.

$AN = x$, $PN = y$, AC the radius $= a$, $AP = s$, (Fig. 6.);

$$\therefore\; y = \sqrt{2ax - x^2},\quad d_x y = \frac{a - x}{\sqrt{2ax - x^2}},$$

$$1 + (d_x y)^2 = \frac{a^2}{2ax - x^2};\quad \therefore\; d_x s = \frac{a}{\sqrt{2ax - x^2}};$$

$$\therefore\; s = a\,\text{versin}^{-1}\frac{x}{a} + C, \text{ and } s = 0 \text{ when } x = 0, \;\therefore\; C = 0;$$

therefore arc $AP = a\,\text{versin}^{-1}\frac{x}{a}$, and making $x = a$, $x = 2a$,

length of quadrant $AD = \frac{\pi a}{2}$, length of semi-circular arc $= \pi a$.

Ex. 4. To find the area of the surface of a sphere.

Let S be the area of the surface generated by the arc AP revolving about AB, (Fig. 6),

$$\therefore\ d_x S = 2\pi y\sqrt{1 + (d_x y)^2} = 2\pi y \frac{a}{\sqrt{2ax - x^2}} = 2\pi a\,;$$

$$\therefore\ S = 2\pi a x + C = 2\pi a \,.\, AN, \text{ since } C = 0\,;$$

and the area of the whole surface $= 2\pi a \,.\, 2a = 4\pi a^2 =$ area of four great circles. Hence the area of the surface generated by $PQ = 2\pi a\,(AN - AM) = 2\pi a \,.\, mn$; that is, if two parallel planes cut a sphere, the area of the intercepted portion of its surface is invariable whatever be their position, provided their distance from one another continue the same; and is equal to the intercepted portion of the surface of the circumscribing cylinder, whose axis is perpendicular to them.

Ex. 5. To find the volume of a sphere.

Let V be the volume generated by the area ANP, (Fig. 6),

$$\therefore\ d_x V = \pi y^2 = \pi(2ax - x^2); \quad \therefore\ V = \pi\left(ax^2 - \frac{x^3}{3}\right), \ (C = 0),$$

or volume of segment, whose height $AN = x$, is $\pi(ax^2 - \frac{1}{3}x^3)$.

Hence volume of $\frac{1}{2}$ sphere $= \dfrac{2\pi a^3}{3}$, making $x = a$; and volume of sphere $= \dfrac{4\pi a^3}{3} = \frac{2}{3}(2\pi a^3) = \frac{2}{3}$ of circumscribing cylinder.

Hence we can express the volume of a frustum by its height and the radii of its ends.

For volume generated by $QMNP = \pi\left(ax^2 - \dfrac{x^3}{3} - ax_1^2 + \dfrac{x_1^3}{3}\right)$

$$= \pi(x - x_1)\left\{a(x + x_1) - \frac{x^2 + xx_1 + x_1^2}{3}\right\}$$

$$= \frac{\pi(x - x_1)}{2}\left\{2a(x + x_1) - x^2 - x_1^2 + \frac{(x - x_1)^2}{3}\right\}.$$

Hence if $QM = b$, $PN = c$, $MN = h$,

volume of frustum (radii of whose ends are b, c, and height h)

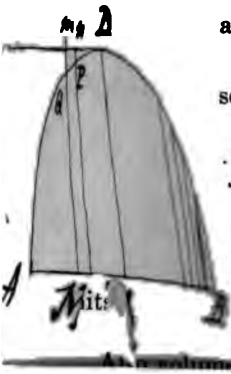
$$= \frac{\pi h}{2}\left\{b^2 + c^2 + \frac{h^2}{3}\right\}.$$

Ex. 6. To find the volume of a spheroid.

pse revolve about its minor axis, the figure gene-
l an *oblate* spheroid; if about its major axis, a
oid.

Fig. 3, making $CM = x$,

and $\therefore (MP)^2 = y^2 = \frac{a^2}{b^2}(b^2 - x^2)$,



segment of an oblate spheroid

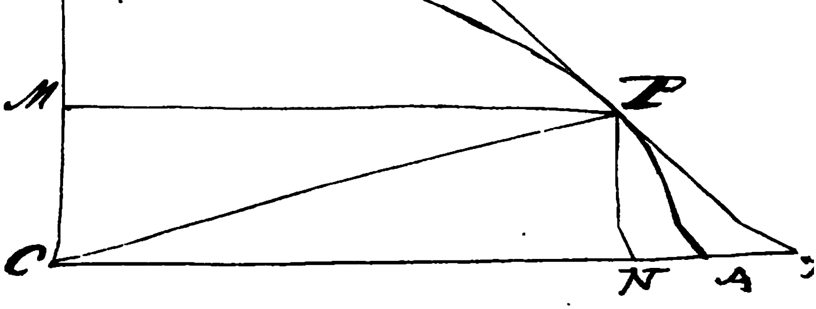
$$\int_x \frac{a^2}{b^2}(b^2 - x^2) = \frac{\pi a^2}{b^2}\left(b^2 x - \frac{x^3}{3}\right) + C;$$

me $= \frac{4\pi a^2 b}{3}$, taking the integral between the

$= b$, and doubling the result.

Also volume of prolate spheroid, making $CN = x$ in Fig. 3.

fig 3



$$\therefore\ s = \sqrt{x^2 + mx} + \frac{m}{2}\log_\epsilon\left(x + \frac{m}{2} + \sqrt{x^2 + mx}\right) + C.$$

But for the arc AP, (Fig. 7.), when $x = 0$, $s = 0$;

$$\therefore\ 0 = \frac{m}{2}\log_\epsilon\left(\frac{m}{2}\right) + C, \text{ or } C = -\frac{m}{2}\log_\epsilon\left(\frac{m}{2}\right);$$

$$\therefore\ \text{arc } AP = \sqrt{x^2 + mx} + \frac{m}{2}\log_\epsilon\left(\frac{2x}{m} + 1 + \frac{2}{m}\sqrt{x^2 + mx}\right). \ ?$$

0 This also admits of the following expression. Let $\angle ASY = PSY = \theta$, AY being a tangent at the vertex, and SY consequently perpendicular to the tangent PY, and bisecting the angle ASP; $\therefore\ SY = m \sec\theta$,

$$\text{hence, Art. 136, arc } AP = m\sec\theta\tan\theta + m\int_\theta \frac{1}{\cos\theta}$$

$$= PY + m\log_\epsilon\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + C, \text{ and } C = 0;$$

$$\text{therefore, arc } AP = PY + AS\,.\,\log_\epsilon\tan\left(\frac{\pi + ASP}{4}\right).$$

Ex. 8. To find the area of a parabolic segment cut off by any ordinate.

Bisect the given ordinate Qq in V (Fig. 8.), and draw PV parallel to the axis AD; draw NM parallel to QV, and let

$$PN = x,\ NM = y,\ p = SP = \frac{m}{(\sin\theta)^2},\ \text{if } \angle\ PTA = \theta,$$

PT being a tangent at P, and, therefore, parallel to Qq; then $y^2 = 4px$, and since the axes of the co-ordinates are inclined to one another at an angle θ,

$$d_xA = y\sin\theta = 2\sin\theta\sqrt{px};\ \therefore\ A = \frac{4}{3}\sin\theta\sqrt{p}x^{\frac{3}{2}} + C;$$

and taking the integral between the limits $x = 0$, $x = PV$, and doubling the result,

$$\text{area } QPq = \frac{8}{3}\sin\theta\sqrt{p}\,(PV)^{\frac{3}{2}} = \frac{2}{3}PV\,.\,Qq\,.\sin\theta$$

$$= \frac{2}{3}\text{ (circumscribing parallelogram } qt\text{)}.$$

Ex. 9. To find the volume of a parabolic frustum, in terms of its height and the radii of its ends.

Let these be $PN = b$, $QM = c$, $NM = h$, (Fig. 7.); it will be found that

$$\text{volume of frustum generated by } PNMQ = \frac{\pi h}{2}(b^2 + c^2).$$

Ex. 10. To find the area of the surface of a paraboloid.

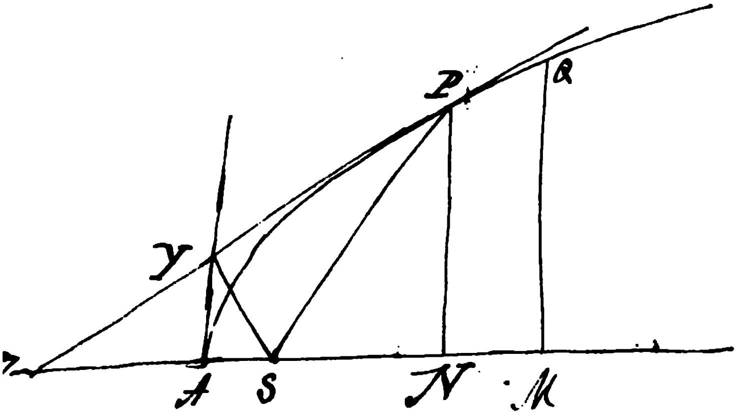
$$d_xS = 2\pi y d_xs = 2\pi\,.\,2\sqrt{mx}\,.\sqrt{1 + \frac{m}{x}} = 4\pi\sqrt{m}\,.\sqrt{x+m}\,;$$

$$\therefore\ S = \frac{8}{3}\pi\sqrt{m}\,(x+m)^{\frac{3}{2}} + C,$$

and since S vanishes when $x = 0$,

$$0 = \frac{8\pi}{3}\sqrt{m}\,.\,m^{\frac{3}{2}} + C\,;$$

$$\therefore \text{ area of surface generated by } AP = \frac{8\pi\sqrt{m}}{3}\left\{(x+m)^{\frac{3}{2}} - m^{\frac{3}{2}}\right\}.$$



$$\therefore\ s = \sqrt{x^2 + mx} + \frac{m}{2} \log_\epsilon \left(x + \frac{m}{2} + \sqrt{x^2 + mx}\right) + C.$$

But for the arc AP, (Fig. 7.), when $x = 0$, $s = 0$;

$$\therefore\ 0 = \frac{m}{2} \log_\epsilon \left(\frac{m}{2}\right) + C, \text{ or } C = -\frac{m}{2} \log_\epsilon \left(\frac{m}{2}\right);$$

$$\therefore\ \text{arc } AP = \sqrt{x^2 + mx} + \frac{m}{2} \log_\epsilon \left(\frac{2x}{m} + 1 + \frac{2}{m}\sqrt{x^2 + mx}\right).$$

0 This also admits of the following expression. Let $\angle ASY = PSY = \theta$, AY being a tangent at the vertex, and SY consequently perpendicular to the tangent PY, and bisecting the angle ASP; $\therefore SY = m \sec\theta$,

$$\text{hence, Art. 136, arc } AP = m \sec\theta \tan\theta + m \int_\theta \frac{1}{\cos\theta}$$

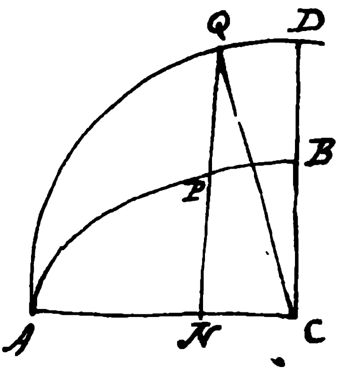
$$= PY + m \log_\epsilon \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + C, \text{ and } C = 0;$$

$$\text{therefore, arc } AP = PY + AS \cdot \log_\epsilon \tan\left(\frac{\pi + ASP}{4}\right).$$

Ex. 8. To find the area of a parabolic segment cut off by any ordinate.

~~Bisect the given ordinate Qq~~ in V (Fig. 8.), and draw PV

fig 5



$$\text{area } QPq = \frac{8}{3}\sin\theta\sqrt{p}\,(PV)^{\frac{3}{2}} = \frac{2}{3}PV.Qq.\sin\theta$$

$$= \frac{2}{3}\text{(circumscribing parallelogram } qt).$$

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Ex. 10. To find the area of the surface of a paraboloid.

$$d_xS = 2\pi y d_x s = 2\pi.2\sqrt{mx}.\sqrt{1+\frac{m}{x}} = 4\pi\sqrt{m}.\sqrt{x+m};$$

$$\therefore\ S = \frac{8}{3}\pi\sqrt{m}\,(x+m)^{\frac{3}{2}} + C,$$

and since S vanishes when $x = 0$,

$$0 = \frac{8\pi}{3}\sqrt{m}.m^{\frac{3}{2}} + C;$$

$$\therefore \text{ area of surface generated by } AP = \frac{8\pi\sqrt{m}}{3}\left\{(x+m)^{\frac{3}{2}} - m^{\frac{3}{2}}\right\}.$$

Ex. 11. To find the length of an elliptic arc.

Let APB be the quadrant of an ellipse, AQD a quadrant of the circumscribed circle, $\angle BCQ = \phi$, QN being an ordinate to the axis, arc $BP = s$; (Fig. 5.),

$$\therefore\ x = CN = a\sin\phi, \text{ and } y = PN = \frac{b}{a}.QN = b\cos\phi;$$

$$\therefore\ (d_\phi s)^2 = (d_\phi x)^2 + (d_\phi y)^2 = (a\cos\phi)^2 + (b\sin\phi)^2$$

$$= a^2\left\{1 - c^2(\sin\phi)^2\right\}, \text{ if } c^2 = \frac{a^2 - b^2}{a^2},$$

$$\text{or } d_{\phi}s = a\sqrt{1 - c^2(\sin\phi)^2};$$

therefore arc BP (the abscissa of whose extremity $= a\sin\phi$) $= a\int_{\phi}\sqrt{1 - c^2(\sin\phi)^2}$, the origin of the integral being $\phi = 0$, which is the fundamental expression for elliptic arcs.

This integral can be obtained only in a series, and by Art. 123,

$$= a\left\{\phi - \frac{c^2}{2}\left(-\tfrac{1}{2}\cos\phi\sin\phi + \frac{\phi}{2}\right) - \frac{1.1}{2.4}c^4\left\{-\tfrac{1}{4}\cos\phi(\sin\phi)^3\right.\right.$$
$$\left.\left. - \frac{1.3}{2.4}\cos\phi\sin\phi + \frac{1.3}{2.4}\phi\right\} - \&\text{c.}\right\}.$$

If we take the integral between the limits $\phi = 0$, $\phi = \dfrac{\pi}{2}$,

$$\text{quadrant } AB = \frac{\pi a}{2}\left\{1 - (\tfrac{1}{2})^2\frac{c^2}{1} - \left(\frac{1.3}{2.4}\right)^2\frac{c^4}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2\frac{c^6}{5} - \&\text{c.}\right\},$$

$$\text{or } = \frac{\pi(a+b)}{4}\left\{1 + (\tfrac{1}{2})^2 n^2 + \left(\frac{1.1}{2.4}\right)^2 n^4 + \left(\frac{1.1.3}{2.4.6}\right)^2 n^6 + \&\text{c.}\right\},$$

$$\text{where } n = \frac{1 - \sqrt{1 - c^2}}{1 + \sqrt{1 - c^2}}, \text{ and } 1 + n = \frac{2a}{a+b}.$$

Of these series for the length of the quadrant, the first is to be used when the eccentricity c is very small; and the second converges rapidly as long as $c^2 < \frac{1}{2}$; in the next Section we shall give theorems for finding the length of any arc to any degree of accuracy.

Ex. 12. To determine two arcs of an ellipse the difference of whose lengths can be expressed by an algebraic quantity.

Let SZ, and $CY = p$, be perpendiculars from the focus and center on the tangent at P, $\angle NPZ = \theta$, (Fig. 3.),

$$\therefore CY^2 = CZ^2 - ZY^2 = CA^2 - CS^2 . (\sin\theta)^2, \text{ or } p = a\sqrt{1 - c^2(\sin\theta)^2};$$

since the locus of Z is a circle, center C, radius CA.

therefore, Art. 136, arc $AP + PY = a\int_\theta \sqrt{1 - c^2 (\sin\theta)^2}$, the origin of the integral being $\theta = 0$; but if $CN' = a \sin\theta$, by the preceding example, arc $BP' = a\int_\theta \sqrt{1 - c^2 (\sin\theta)^2}$, the origin of the integral being $\theta = 0$;

$$\therefore \text{ arc } AP + PY = \text{arc } BP'.$$

Let $CN = a \sin\phi$, $\therefore PN = b \cos\phi$;

$$\therefore \cot\theta = -d_x y = \frac{b \sin\phi}{a \cos\phi} = \frac{b}{a} \tan\phi, \text{ or } \cot\theta \cot\phi = \sqrt{1 - c^2};$$

$$\text{also } PY = -d_\theta p = \frac{ac^2 \sin\theta \cos\theta}{\sqrt{1 - c^2 (\sin\theta)^2}} = ac^2 \sin\theta \sin\phi.$$

$$\text{Hence arc } BP' - \text{arc } AP = ac^2 \sin\theta \sin\phi = \frac{c^2}{a} CN' . CN,$$

the abscissæ of the points P, P', viz. $a \sin\phi$, $a \sin\theta$, being connected by the equation $\cot\phi \cot\theta = \sqrt{1 - c^2}$. This is Fagnani's theorem.

Since ϕ and θ are similarly involved in the value of PY, we have $PY = P'Y'$, or the two points P, P', are such, that the intercepts of the tangents are equal to one another.

Ex. 13. To determine a point which shall divide an elliptic quadrant into two parts, the difference of whose lengths shall be equal to the difference of the semi-axes.

Let $\phi = \theta$, $\therefore (\cot\theta)^2 = \frac{b}{a}$; then P, P', coincide in a fixed point K whose co-ordinates are

$$a \sin\theta = a \sqrt{\frac{a}{a + b}}, \quad b \cos\theta = b \sqrt{\frac{b}{a + b}},$$

$$\text{and } BK - AK = ac^2 (\sin\theta)^2 = a - b.$$

Also $CK = \sqrt{a^2 - ab + b^2}$, and the $\frac{1}{2}$ conjugate diameter $= \sqrt{ab}$.

The points P and P' being so related that if $N'P'$ were produced to meet the circle on the major axis in Q', and CQ' joined, $NPT = BCQ'$.

Ex. 14. To find the length of a hyperbolic arc.

Let SZ, and $CY = p$, be perpendiculars from the center and focus on the tangent at P; $CS = a$, $CA = ac$, $\angle NPT = \theta$, (Fig. 4.),

$$\therefore\ CY^2 = CZ^2 - ZY^2 = CA^2 - CS^2 (\sin\theta)^2, \text{ or } p = a\sqrt{c^2 - (\sin\theta)^2};$$

$$\therefore\ PY - \text{arc } AP = a \int_\theta \sqrt{c^2 - (\sin\theta)^2},$$

the origin of the integral being $\theta = 0$.

Now $\angle PTN$ has its least value when the tangent coincides with the asymptote, in which case its cosine $= CA \div CS = c$;

$\therefore \sin\theta$, which $= \cos PTN$, cannot exceed c; let $\therefore \sin\theta = c \sin\phi$,

$$\therefore\ \int_\theta \sqrt{c^2 - (\sin\theta)^2} = \int_\phi c\cos\phi\, d_\phi\theta = \int_\phi \frac{c^2 (\cos\phi)^2}{\sqrt{1 - c^2 (\sin\phi)^2}}$$

$$= \int_\phi \sqrt{1 - c^2 (\sin\phi)^2} - (1 - c^2) \int_\phi \frac{1}{\sqrt{1 - c^2 (\sin\phi)^2}},$$

$$\text{and } PY = -d_\theta p = \frac{a \sin\theta \cos\theta}{\sqrt{c^2 - (\sin\theta)^2}} = a \tan\phi \sqrt{1 - c^2 (\sin\phi)^2},$$

$$\therefore\ \text{arc } AP = a \tan\phi \sqrt{1 - c^2 (\sin\phi)^2} - a \int_\phi \sqrt{1 - c^2 (\sin\phi)^2}$$

$$+\, a (1 - c^2) \int_\phi \frac{1}{\sqrt{1 - c^2 (\sin\phi)^2}},$$

the origin of both integrals being $\phi = 0$; and if we substitute for them, their values derived from Art. 122, we shall have an expression for the arc AP in a series ascending by powers of c which is always < 1.

Ex. 15. To find the difference between the lengths of the asymptote and infinite hyperbolic arc.

When $\phi = \frac{\pi}{2}$, $p = 0$, that is, the tangent passes through the centre and coincides with the asymptote. Hence the difference between the lengths of the asymptote and the infinite hyperbolic arc

$$= \left(\int_{\phi=\frac{\pi}{2}} - \int_{\phi=0}\right) \frac{ac^2(\cos\phi)^2}{\sqrt{1-c^2(\sin\phi)^2}} = \frac{\pi a}{2}\left\{1-(\tfrac{1}{2})^2\frac{c^2}{1} - \left(\frac{1.3}{2.4}\right)^2\frac{c^4}{3} - \&c.\right\}$$

$$-\frac{\pi a(1-c^2)}{2}\left\{1+(\tfrac{1}{2})^2 c^2 + \left(\frac{1.3}{2.4}\right)^2 c^4 + \&c.\right\}.$$

In the next Section we shall obtain more convenient expressions both for this difference, and for the length of any arc of an hyperbola.

Ex. 16. To find the area of the surface of a prolate spheroid.

In this case the arc BP revolves about AC; Fig. 3.

$$\therefore\ d_\phi S = 2\pi PN d_\phi BP = 2\pi b\cos\phi . a\sqrt{1-c^2(\sin\phi)^2},\ \text{(by Ex. 11.)}$$

$$= \frac{2\pi ab}{c} d_\phi (c\sin\phi)\sqrt{1-(c\sin\phi)^2},$$

$$\therefore\ S = \frac{2\pi ab}{c}\left\{\tfrac{1}{2}c\sin\phi\sqrt{1-(c\sin\phi)^2} + \tfrac{1}{2}\sin^{-1}(c\sin\phi)\right\} + C,$$

and $C = 0$, because S and ϕ vanish together,

$\therefore$ area of surface generated by arc BP

$$= \frac{\pi ab}{c}\left\{c\sin\phi\sqrt{1-c^2(\sin\phi)^2} + \sin^{-1}(c\sin\phi)\right\};$$

and making $\phi = \frac{\pi}{2}$, and multiplying by 2,

$$\text{area of whole surface} = \frac{2\pi ab}{c}\left(c\sqrt{1-c^2} + \sin^{-1}c\right).$$

If c be very small, the area of the surface

$$= \frac{2\pi ab}{c}\left\{c\left(1-\frac{c^2}{2}\right) + \left(c+\frac{c^3}{6}\right)\right\} = 4\pi ab\left(1-\frac{c^2}{6}\right)$$

$$= 4\pi a^2\left(1-\frac{1}{2}c^2\right)\left(1-\frac{1}{6}c^2\right) = 4\pi a^2\left(1-\frac{2}{3}c^2\right).$$

Ex. 17. To find the area of the surface of an oblate spheroid.

In this case the arc AP revolves about BC; Fig. 3.

$$\therefore\ d_\phi S = 2\pi CN d_\phi AP = -2\pi a \sin\phi \,.\, a\sqrt{1-(c\sin\phi)^2}$$

$$= 2\pi\frac{a^2}{c} d_\phi (c\cos\phi)\sqrt{1-c^2+c^2(\cos\phi)^2};$$

$$\therefore\ S = \frac{2\pi a^2}{c}\left\{\frac{c\cos\phi}{2}\sqrt{1-c^2+(c\cos\phi)^2}\right.$$

$$\left. + \frac{1-c^2}{2}\log_\epsilon\left\{c\cos\phi+\sqrt{1-c^2+(c\cos\phi)^2}\right\}\right\} + C,$$

$$\text{and } S = 0, \text{ when } \phi = \frac{\pi}{2},$$

$$\therefore\ 0 = \frac{2\pi a^2}{c}\left(\frac{1-c^2}{2}\log_\epsilon\sqrt{1-c^2}\right) + C;\ \therefore \text{ eliminating } C,$$

area of surface generated by arc $AP = \dfrac{\pi a^2}{c}\left\{c\cos\phi\sqrt{1-(c\sin\phi)^2}\right.$

$$\left. + (1-e^2)\log_\epsilon\left(\frac{c\cos\phi+\sqrt{1-(c\sin\phi)^2}}{\sqrt{1-c^2}}\right)\right\};$$

and area of whole surface $= \dfrac{2\pi a^2}{c}\left\{c + (1-c^2)\log_\epsilon\sqrt{\dfrac{1+c}{1-c}}\right\}$.

If c be very small, area of whole surface

$$= \frac{2\pi a^2}{c}\left\{c + (1-c^2)\left(c+\frac{c^3}{3}\right)\right\} = 4\pi a^2\left(1-\frac{c^2}{3}\right).$$

Ex. 18. To find the area of a cycloid.

Let AB the axis $= 2a$, $AN = x$, $PN = y$, (Fig. 9.), then $PQ =$ circular arc AQ, by property of the cycloid,

$$\therefore\ y = AQ + QN = a \text{ versin}^{-1}\frac{x}{a} + \sqrt{2ax-x^2},$$

$$0 = \pi a^3\left\{2 - \tfrac{1}{4} - 1 + \tfrac{1}{3}\right\} + C, \quad \therefore\ C = -\pi a^3 \cdot \frac{13}{12};$$

therefore, making $\theta = \pi$,

$$\text{whole volume} = \pi a^3\left\{\pi^2 - 2 + \tfrac{1}{2}\pi^2 - \tfrac{1}{4} + 1 - \tfrac{1}{3} - \frac{13}{12}\right\}$$

$$= \pi a^3\left\{\frac{3\pi^2}{2} - \frac{8}{3}\right\}.$$

If the cycloid revolve about its base, it may be shewn that the volume generated $= 5\pi^2 a^3$.

Ex. 21. To find the area of the surface of the figure generated by the revolution of a cycloid about its axis. (Fig. 9.)

$$S = 2\pi \int_x y\, d_x s = 2\pi\left(ys - \int_x s\, d_x y\right)$$

$$= 2\pi\left(ys - \int_x 2\sqrt{2ax}\sqrt{\frac{2a - x}{x}}\right)$$

$$= 2\pi\left\{ys + \frac{4}{3}\sqrt{2a}\,(2a - x)^{\frac{3}{2}}\right\} + C,$$

$$0 = 2\pi\left(\frac{4}{3} \cdot 4a^2\right) \qquad + C,$$

therefore area of surface generated by AP

$$= 2\pi\left\{ys + \frac{4}{3}\sqrt{2a}\,(2a - x)^{\frac{3}{2}} - \frac{16}{3}a^2\right\};$$

and making $x = AB = 2a$, and therefore $y = \pi a$, $s = 4a$,

$$\text{area of whole surface} = 2\pi\left(4\pi a^2 - \frac{16}{3}a^2\right) = 8\pi a^2\left(\pi - \frac{4}{3}\right).$$

If the cycloid revolve round its base, it may be shewn that the area of the surface of the figure generated $= \dfrac{64\pi a^2}{3}$.

Ex. 22. To find the length of the Epitrochoid, which is generated by a fixed point in the plane of a circle whose circumference rolls along the outside of the circumference of another circle.

Let the radius of the fixed circle $CA = a$, the radius of generating circle $Ba = b$, (Fig. 14), and the distance of the describing point from the center of the generating circle $BP = c$. A, a, points in contact at first, $CN = x$, $PN = y$, arc $Pp = s$, $\angle ACB = \theta$. Draw Pq parallel to CA,

$$\therefore \ \angle BPq = PBQ + PQB = \frac{a\theta}{b} + \theta;$$

$$\therefore \ x = CB \cos ACB - BP \cos BPq = (a+b)\cos\theta - c\cos\left(\frac{a}{b} + 1\right)\theta,$$

$$y = CB \sin ACB - BP \sin BPq = (a+b)\sin\theta - c\sin\left(\frac{a}{b} + 1\right)\theta;$$

$$\therefore \ (d_\theta s)^2 = (d_\theta x)^2 + (d_\theta y)^2 = (a+b)^2 + c^2\left(\frac{a}{b} + 1\right)^2$$

$$-2\frac{c}{b}(a+b)^2\cos\frac{a\theta}{b}; \ \therefore \ d_\theta s = (a+b)\left(1 + \frac{c^2}{b^2} - 2\frac{c}{b}\cos\frac{a\theta}{b}\right)^{\frac{1}{2}}$$

$$= (a+b)\left(1 + \frac{c}{b}\right)\left\{1 - \frac{4bc}{(b+c)^2}\left(\cos\frac{a\theta}{2b}\right)^2\right\}^{\frac{1}{2}};$$

$$\text{or } d_\phi s = -2\left(1 + \frac{b}{a}\right)(b+c)\sqrt{1 - n^2(\sin\phi)^2},$$

$$\text{making } \phi = \frac{\pi}{2} - \frac{a\theta}{2b}, \text{ and } n^2 = \frac{4bc}{(b+c)^2}.$$

Hence s can be expressed by an elliptic arc.

To get the length of the arc traced out by half a revolution of the generating circle, we must integrate between the limits $\theta = 0$, $\frac{a\theta}{b} = \pi$, or $\phi = \frac{\pi}{2}$, $\phi = 0$;

$$\therefore \ s = \pi\left(1 + \frac{b}{a}\right)(b+c)\left\{1 - \left(\tfrac{1}{2}\right)^2\frac{n^2}{1} - \left(\frac{1.3}{2.4}\right)^2\frac{n^4}{3} - \&c.\right\}.$$

If $c = b$, or the describing point be in the circumference, the curve is the epicycloid, and $d_\theta s = 2(a+b)\sin\frac{a\theta}{2b}$,

$$\therefore\ s = -\frac{4b(a+b)}{a}\cos\frac{a\theta}{2b} + C, \text{ and } s = 0 \text{ when } \theta = 0,$$

$$\therefore\ \text{arc}\, pP = \frac{4b(a+b)}{a}\left(1 - \cos\frac{a\theta}{2b}\right) = \frac{4b(a+b)}{a}\text{ versin }\frac{PBQ}{2};$$

therefore arc described by one revolution of the generating circle $= \frac{8b(a+b)}{a}$.

Of course the above formulæ are adapted to the hypotrochoid and hypocycloid (for which the circle rolls along the *inside* of the circumference of the fixed circle) by making b negative.

Ex. 23. To find the area of a catenary.

Let B be the origin, $AB = a$, $BN = x$, $PN = y$, (Fig. 10.),

$$\therefore\ \frac{y}{a} = \tfrac{1}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right);$$

$$\therefore\ \text{area } ABPN = \textstyle\int_x y = \int_x \frac{a^2}{2}\left(\frac{1}{a}e^{\frac{x}{a}} + \frac{1}{a}e^{-\frac{x}{a}}\right)$$

$$= \frac{a^2}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right), \text{ (const. } = 0).$$

Ex. 24. To find the area of the Cissoid of Diocles.

This curve is the locus of the intersection of an ordinate of a circle, with a line joining the extremity of another equal ordinate and the extremity of the diameter to which both are perpendicular.

Let QN, $Q'N'$, be the equal ordinates, and P a point in the curve.

$$AB = 2a,\ AN = x,\ PN = y, \text{ (Fig. 11.)},$$

$$\therefore \frac{PN^2}{AN^2} = \frac{Q'N'^2}{AN'^2} = \frac{x}{2a - x}, \quad \therefore y^2 = \frac{x^3}{2a - x};$$

$$\therefore \text{area } ANP = \int_x \frac{x^{\frac{3}{2}}}{\sqrt{2a - x}} = -2x^{\frac{3}{2}}\sqrt{2a - x} + 3\int_x \sqrt{2ax - x^2}$$

$$= -2x\sqrt{2ax - x^2} + 3\,(\text{circular area } ANQ)\,; \quad (\text{const.} = 0)\,;$$

therefore, making $x = 2a$, whole area contained between the curve and its asymptote $= \dfrac{3\pi a^2}{2} = 3$ times semicircle AQB.

Also the volume of the figure generated by the revolution of the area ANP about the axis AB

$$= \left(\int_x - \int_{x=0}\right) \frac{\pi x^3}{2a - x} = \pi \left(-\frac{x^3}{3} - ax^2 - 4a^2x + 8a^3 \log_\epsilon \frac{2a}{2a - x}\right),$$

2

by dividing and integrating.

b

If the figure revolve about the asymptote,

$$d_x V = \pi\,(PM)^2 d_x BM = \pi\,(2a - x)^2 \,.\, \frac{(3a - x)\sqrt{x}}{(2a - x)^{\frac{3}{2}}}$$

$$= \pi(3a - x)\sqrt{2ax - x^2} = \pi\left\{2a\sqrt{2ax - x^2} + (a - x)\sqrt{2ax - x^2}\right\},$$

$$\therefore \quad V = \pi\left\{2a\,(\text{circular area } ANQ) + \tfrac{1}{3}(2ax - x^2)^{\frac{3}{2}}\right\}, \ (\text{const.} = 0)\,;$$

$$\therefore \text{ whole volume } = \pi\left(2a \,.\, \frac{\pi a^2}{2}\right) = \pi^2 a^3.$$

The curve has another branch similar to APD situated below AB.

Ex. 25. To find the volume of the figure generated by the revolution of a Conchoid about its asymptote.

This curve is described by taking PD always of the same length, AN being a line given in position, and C a fixed point about which CP revolves. Draw CB perpendicular to AN, and let

$$AN = x, \quad PN = y, \quad AC = a, \quad AB = DP = b; \quad (\text{Fig. } 12.),$$

$$\text{then } \frac{PN^2}{DN^2} = \frac{CM^2}{PM^2} \text{ or } \frac{y^2}{b^2 - y^2} = \frac{(a+y)^2}{x^2},$$

$$\therefore\ x = \frac{a+y}{y}\sqrt{b^2 - y^2},\ d_y x = -\frac{ab^2}{y^2\sqrt{b^2 - y^2}} - \frac{y}{\sqrt{b^2 - y^2}},$$

$$\therefore\ d_y V = \pi y^2 d_y x = -\pi\left(\frac{ab^2}{\sqrt{b^2 - y^2}} + \frac{y^3}{\sqrt{b^2 - y^2}}\right)$$

$$= \pi\left(-\frac{ab^2}{\sqrt{b^2 - y^2}} + y\sqrt{b^2 - y^2} - b^2\frac{y}{\sqrt{b^2 - y^2}}\right),$$

$$\therefore\ V = \pi\left\{ab^2 \cos^{-1}\frac{y}{b} - \tfrac{1}{3}(b^2 - y^2)^{\frac{3}{2}} + b^2\sqrt{b^2 - y^2}\right\} + C,$$

and $V = 0$ when $y = b$, $\therefore\ C = 0$,

therefore, making $y = 0$, whole volume

$$= \pi\left(ab^2 . \frac{\pi}{2} + \tfrac{2}{3}b^3\right) = \pi b^2\left(\frac{\pi a}{2} + \frac{2b}{3}\right).$$

The inferior Conchoid is traced out by taking in DC, $DP' = DP = b$; the volume generated by it will result from the above expression by making b negative, provided $b < a$.

138. We shall next exemplify the use of polar co-ordinates in finding the areas and lengths of curves.

If s be the length of the arc of a curve referred to polar co-ordinates, intercepted between the radius vector ρ and a fixed line, and A the sectorial area bounded by these and the arc, also θ the angle made by ρ with the initial line, then

$$d_\theta s = \sqrt{\rho^2 + (d_\theta \rho)^2},\ d_\theta A = \frac{\rho^2}{2},\ \therefore\ s = \int_\theta \sqrt{\rho^2 + (d_\theta \rho)^2},\ A = \tfrac{1}{2}\int_\theta \rho^2.$$

These formulæ are to be used when ρ is given in terms of θ; if θ be given in terms of ρ, the formulæ become

$$s = \int_\rho \sqrt{\rho^2 (d_\rho \theta)^2 + 1},\quad A = \tfrac{1}{2}\int_\rho \rho^2 d_\rho \theta.$$

Since $\tan\theta = \dfrac{y}{x}$,

$$\therefore\ d_x A = \tfrac{1}{2}\rho^2 d_x\theta = \tfrac{1}{2}\rho^2 d_x\left(\tan^{-1}\frac{y}{x}\right) = \tfrac{1}{2}(x d_x y - y);$$

$$\therefore\ A = \tfrac{1}{2}\int_x (x d_x y - y).$$

This expression, in which x is independent variable, is sometimes convenient for finding a sectorial area.

In some cases it is advantageous to use expressions involving p, the perpendicular dropped from the pole upon the tangent;

in such cases, since $d_\rho s = \dfrac{\rho}{\sqrt{\rho^2 - p^2}}$, $d_\rho A = \tfrac{1}{2}\dfrac{p\rho}{\sqrt{\rho^2 - p^2}}$,

we have $s = \displaystyle\int_\rho \frac{\rho}{\sqrt{\rho^2 - p^2}}$, $A = \tfrac{1}{2}\displaystyle\int_\rho \frac{p\rho}{\sqrt{\rho^2 - p^2}}$.

Ex. 1. To find the area of the Conchoid.

Let $CP = \rho$, $\angle\ BCP = \theta$, (Fig. 12.),

then $CP = CD + DP = CD + AB$, or $\rho = a\sec\theta + b$;

therefore, area $BCP = \tfrac{1}{2}\int_\theta \rho^2 = \tfrac{1}{2}\int_\theta \{a^2(\sec\theta)^2 + 2ab\sec\theta + b^2\}$

$$= \tfrac{1}{2}\left\{a^2\tan\theta + 2ab\log_\epsilon\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + b^2\theta\right\},\ (\text{const.} = 0).$$

Ex. 2. To find the area of any sector of a hyperbola bounded by straight lines passing through the center.

Let C be the center of the hyperbola, a, b, the semi-axes, $CN = x$, $PN = y$, $CP = \rho$, $ACP = \theta$, (Fig. 4).

If the hyperbola be rectangular or $a = b$,

$$x^2 - y^2 = a^2,\ \text{or}\ \rho^2(\cos\theta)^2 - \rho^2(\sin\theta)^2 = a^2,\ \therefore\ \rho^2 = \frac{a^2}{\cos 2\theta};$$

$$\therefore\ \text{area } ACP = \tfrac{1}{2}a^2\int_\theta \frac{1}{\cos 2\theta} = \tfrac{1}{4}a^2\log_\epsilon\left\{\tan\left(\theta + \frac{\pi}{4}\right)\right\},\ (\text{const.} = 0).$$

If the hyperbola be not rectangular,

$$d_x(\text{area } ACP) = \tfrac{1}{2}(x d_x y - y)$$

$$= \tfrac{1}{2}\left(\frac{b}{a}\frac{x^2}{\sqrt{x^2-a^2}} - \frac{b}{a}\sqrt{x^2-a^2}\right) = \frac{ab}{2}\frac{1}{\sqrt{x^2-a^2}},$$

$$\therefore \text{ area } ACP = \frac{ab}{2}\log_e\left(\frac{x}{a} + \frac{1}{a}\sqrt{x^2-a^2}\right)$$

$$= \frac{ab}{2}\log_e\left(\frac{x}{a} + \frac{y}{b}\right).$$

Hence if we put the sectorial area $ACP = \frac{ab}{2}\theta$, we may express the co-ordinates of P in terms of θ;

$$\text{for then } \frac{x}{a} + \frac{y}{b} = e^{\theta}, \ \frac{x}{a} - \frac{y}{b} = e^{-\theta}, \text{ since } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;$$

$$\therefore \ x = \frac{a}{2}(e^{\theta} + e^{-\theta}), \ y = \frac{b}{2}(e^{\theta} - e^{-\theta}).$$

Ex. 3. To find the area of a hyperbolic sector bounded by straight lines passing through the focus.

Let $CA = a$, $CS = ae$, $CN = x$, $PN = y$, (Fig. 4.),

also because x cannot be less than a, let $x = a\sec\theta$, and therefore $y = b\tan\theta$;

$$\text{now area } ASP = \text{area } ANP + \text{area of } \Delta SNP,$$

$$\therefore \ d_x(\text{area } ASP) = y + \tfrac{1}{2}d_x\{(ae-x)\cdot y\} = \tfrac{1}{2}\{y + (ae-x)d_x y\};$$

$$\therefore \ d_\theta(\text{area } ASP) = \tfrac{1}{2}\{y d_\theta x + (ae-x)d_\theta y\}$$

$$= \tfrac{1}{2}\{ab(\tan\theta)^2\sec\theta + ab(e-\sec\theta)(\sec\theta)^2\}$$

$$= \frac{ab}{2}\{e(\sec\theta)^2 - \sec\theta\};$$

$$\therefore \text{ area } ASP = \frac{ab}{2}\left\{e \tan\theta - \log_\epsilon \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right\}, \text{ (const.} = 0);$$

for the area = 0, when $x = a$, and therefore $\theta = 0$.

It may easily be shewn that if

$$\angle ASP = v, \ \tan\frac{v}{2} = \sqrt{\frac{e+1}{e-1}}\tan\frac{\theta}{2}.$$

Ex. 4. To find the area and length of the Lemniscata, which is the locus of the intersection of the tangent to a rectangular hyperbola and the perpendicular upon it from the center.

Let C and S be the center and focus of a rectangular hyperbola, (fig. 13.) therefore $CS = CA\sqrt{2}$; CP, SZ, perpendiculars upon a tangent, $CP = \rho$, $ACP = \theta$, $AC = a$;

$$\text{then } CP^2 = CZ^2 - ZP^2 = CA^2 - CS^2(\sin\theta)^2 = a^2\{1 - 2(\sin\theta)^2\},$$

$$\text{or } \rho^2 = a^2\cos 2\theta,$$

the equation to the curve, which is manifestly of the form represented, symmetrical with respect to the line AA';

$$\therefore \text{ area } ACP = \frac{a^2}{2}\textstyle\int_\theta \cos 2\theta = \frac{a^2}{4}\sin 2\theta, \quad \text{(const.} = 0).$$

Make $\theta = \frac{1}{4}\pi$, then $\rho = 0$, and area $APBC = \dfrac{a^2}{4}$,

therefore whole area = a^2.

$$\text{Also arc } AP = \textstyle\int_\theta \sqrt{\rho^2 + (d_\theta \rho)^2} = a\displaystyle\int_\theta \frac{1}{\sqrt{\cos 2\theta}}$$

$$= \frac{a}{\sqrt{2}}\int_\phi \frac{1}{\sqrt{1 - \frac{1}{2}(\sin\phi)^2}}, \text{ (making } \sqrt{\cos 2\theta} = \cos\phi),$$

which may be integrated in a series by Art. 125.

Ex. 5. To find the area of the curve traced out by the intersection of normals to an ellipse which are at right angles to one another.

Its equation is $\rho = c\dfrac{1-(n\tan\theta)^2}{1+(n\tan\theta)^2}$, and the whole area

$$= \pi c^2 \frac{1+n^2}{(1+n)^2} = \pi(a-b)^2, \text{ where } c = \frac{a^2-b^2}{\sqrt{a^2+b^2}}, \text{ and } n = \frac{a}{b},$$

a and b being the semi-axes of the ellipse.

Ex. 6. To find the length and area of a Hypocycloid.

CF, the radius of the fixed circle, $= a$, (Fig. 15.) GF the diameter of the rolling circle $= 2b$, P the generating point at first in contact with B, BPE the curve traced out in half a revolution; $CP = \rho$, CY, a perpendicular upon the tangent at P, $= p$. It is evident that when the rolling circle is turning upon F, the motion of P is perpendicular to FP, and, therefore, the tangent at P passes through G; let $CG = a - 2b = c$, then by similar triangles

$$\left(\frac{PY}{CF}\right)^2 = \left(\frac{GY}{CG}\right)^2, \text{ or } \frac{\rho^2 - p^2}{a^2} = \frac{c^2 - p^2}{c^2}; \therefore p^2 = c^2 \frac{a^2-\rho^2}{a^2-c^2};$$

$$\therefore d_\rho EP = \frac{\rho}{\sqrt{\rho^2 - p^2}} = \sqrt{1 - \frac{c^2}{a^2}} \cdot \frac{\rho}{\sqrt{\rho^2 - c^2}},$$

$$\therefore EP = \frac{1}{a}\sqrt{a^2 - c^2} \cdot \sqrt{\rho^2 - c^2},$$

for $EP = 0$ when $\rho = c$, and $\therefore$ const. $= 0$.

Hence making $\rho = a$, and doubling the result,

DEB, the arc traced out in one revolution, $= \dfrac{2(a^2-c^2)}{a} = 8b\left(1 - \dfrac{b}{a}\right)$; this becomes $= 8b$, when a is infinite, as it ought, for the curve is then the common cycloid.

It may be easily shewn that arc $BP = \dfrac{4b(a-b)}{a}\,\text{versin}\,\dfrac{FOP}{2}$.

Again, $d_\rho\,(\text{area } ECP) = \frac{1}{2}\frac{p\rho}{\sqrt{\rho^2-p^2}} = \frac{c}{2a}\rho\sqrt{\frac{a^2-\rho^2}{\rho^2-c^2}}$

$$= \frac{c}{2a}\,d_\rho z\,.\,\sqrt{a^2-c^2-z^2}, \text{ if } z=\sqrt{\rho^2-c^2};$$

$$\therefore \text{ area } ECP = \frac{c}{2a}\left\{\frac{z}{2}\sqrt{a^2-c^2-z^2}+\frac{a^2-c^2}{2}\sin^{-1}\frac{z}{\sqrt{a^2-c^2}}\right\}+C,$$

$$= \frac{c}{4a}\left\{\sqrt{\rho^2-c^2}\,.\,\sqrt{a^2-\rho^2}+(a^2-c^2)\sin^{-1}\sqrt{\frac{\rho^2-c^2}{a^2-c^2}}\right\}$$

(const. $=0$, for area $=0$, when $\rho=c$); therefore, making $\rho=a$, and multiplying by 2, area $BCDE = \frac{c}{4a}\,.\,(a^2-c^2)\,\pi$,

$$\text{and area } BADE = -\frac{c}{4a}(a^2-c^2)\,\pi+\pi ab = \pi b^2\left(3-\frac{2b}{a}\right).$$

Ex. 7. To find the length and area of the involute of a circle.

Let AP (Fig. 16.) be described by unwinding the string PT kept constantly stretched from the arc AT; then PT is manifestly perpendicular to the curve, and therefore if PY be a tangent, and CY a perpendicular upon it from the center, TY is a rectangle, and consequently $p=\sqrt{\rho^2-a^2}$, making $CP=\rho$, $CY=p$, $CA=a$. Let $AP=s$,

$$\therefore\ d_\rho s = \frac{\rho}{\sqrt{\rho^2-p^2}} = \frac{\rho}{a},\quad \therefore\ s=\frac{\rho^2}{2a}+C, \text{ and } 0=\frac{a}{2}+C,$$

$$\therefore\ s=\frac{\rho^2-a^2}{2a}=\frac{p^2}{2a}=\frac{1}{2}\frac{(AT)^2}{a};$$

if, therefore, the string be unwound from n circumferences, the length of P's path $=\frac{1}{2}\frac{(2n\pi a)^2}{a}=2n^2\pi^2 a$.

Also area $ACP = \frac{1}{2}\int_\rho \frac{p\rho}{\sqrt{\rho^2 - p^2}} = \frac{1}{2a}\int_\rho \rho\sqrt{\rho^2 - a^2}$

$$= \frac{(\rho^2 - a^2)^{\frac{3}{2}}}{6a} = \frac{p^3}{6a}.$$

It is manifest that the circular sector ACT equals the triangular area PCT; therefore, taking each of these separately from the area $PTCA$, we have area PAT = area PAC; that is, the area between the curve and its evolute, is equal to the area swept out by the radius vector.

139. The area of the nodus of a curve may be found by polar co-ordinates, as we have seen in some of the preceding Examples. It is sometimes, however, convenient to make t the tangent of the spiral angle the independent variable; in that case, since

$$d_\theta A = \tfrac{1}{2}\rho^2,\quad d_t A = \tfrac{1}{2}\rho^2 d_t\theta = \tfrac{1}{2}\rho^2 (\cos\theta)^2 = \frac{x^2}{2},\quad \therefore\ A = \tfrac{1}{2}\int_t x^2.$$

Hence since $y = xt$, if we express x in terms of t from the equation to the curve, and integrate between proper limits, we shall have the area of the corresponding nodus.

Ex. 1. $y^3 - 3axy + x^3 = 0$, Fig. 17.

If the curve, of which this is the equation, be traced, there will be found a nodus to which the axes of x and y are tangents.

$$\text{Let } y = xt,\ \therefore\ t^3 - \frac{3at}{x} + 1 = 0,\ \text{or } x = \frac{3at}{1 + t^3};$$

therefore $x = 0$ when $t = 0$, and also when $t = \infty$;

$$\therefore\ A = \frac{9a^2}{2}\int_t \frac{t^2}{(1 + t^3)^2} = C - \frac{3a^2}{2}\frac{1}{1 + t^3};\ \text{make } t = 0,$$

$$\therefore\ 0 = C - \frac{3a^2}{2};\ \text{make } t = \infty\text{, and area } ABD = C = \frac{3a^2}{2}.$$

Ex. 2. $(x^2 + y^2)^2 = (ax)^2 - (by)^2$, (Fig. 13.);

$$\text{whole area} = ab + (a^2 - b^2)\tan^{-1}\frac{a}{b}.$$

Ex. 3. $y^n = ax^{n-2}(x - my)$;

$$\text{area of nodus} = \frac{a^2 m^{2n-1}}{(2n-1)(2n-2)(2n-3)}.$$

Ex. 4. To find the area of the evolute of an ellipse.

The equation is $\left(\dfrac{x}{\alpha}\right)^{\frac{2}{3}} + \left(\dfrac{y}{\beta}\right)^{\frac{2}{3}} = 1$; where $\alpha = \dfrac{a^2-b^2}{a}$, $\beta = \dfrac{a^2-b^2}{b}$; a and b being the semi-axes of the ellipse.

$$\text{Make } y = xt; \ \therefore \ x^{\frac{2}{3}} = \frac{\alpha^{\frac{2}{3}}}{1 + (nt)^{\frac{2}{3}}}, \text{ where } n = \frac{\alpha}{\beta};$$

$$\therefore \ d_t A = \frac{x^2}{2} = \frac{\alpha^2}{2} \cdot \frac{1}{\{1 + (nt)^{\frac{2}{3}}\}^3}; \text{ let } (nt)^{\frac{2}{3}} = z^2,$$

$$\therefore \ d_z A = \frac{3\alpha^2}{2n} \cdot \frac{z^2}{(1+z^2)^3} = \frac{3\alpha\beta}{2} \cdot \left(\frac{1}{(1+z^2)^2} - \frac{1}{(1+z^2)^3}\right);$$

the integral must be taken between the limits $t = 0$, $t = \infty$,

or $z = 0$, $z = \infty$;

$$\therefore \ A = \frac{3\alpha\beta}{2}\left(\frac{1}{2} - \frac{3 \cdot 1}{4 \cdot 2}\right)\frac{\pi}{2} = \frac{3\alpha\beta}{2}\,\frac{\pi}{16};$$

$$\text{therefore the whole area} = \frac{3\pi}{8} \cdot \frac{(a^2 - b^2)^2}{ab}.$$

Also the length of a quadrant of the evolute = difference of the radii of curvature of the ellipse at its extremities,

$$= \frac{a^2}{b} - \frac{b^2}{a} = \frac{a^3 - b^3}{ab};$$

$$\text{therefore whole length of evolute} = \frac{4}{ab}(a^3 - b^3).$$

140. In the following examples, the revolving area is bounded by two curves, instead of a curve and the axis.

Ex. 1. To find the volume of the figure generated by the revolution of the curve whose equation is

$$y^4 - 2axy^2 - 3a^2x^2 + x^4 = 0,$$

about the axis of x.

The equation, solved with respect to y^2, is

$$y^2 = ax \pm x\sqrt{4a^2 - x^2},$$

and its form is seen (Fig. 18.), the lower sign producing the portion DBD', and the upper sign the portion DAD'.

When x is negative, the lower sign only is admissible, and gives the oval AE; where $AE = AB = a\sqrt{3}$, and $AF = 2a$.

Hence volume generated by ACB

$$= \left(\int_{x=a\sqrt{3}} - \int_{x=0}\right) \pi \left(ax + x\sqrt{4a^2 - x^2}\right) = \frac{23\pi a^3}{6};$$

volume generated by CDB

$$= \left(\int_{x=2a} - \int_{x=a\sqrt{3}}\right) \pi \left(2x\sqrt{4a^2 - x^2}\right) = \frac{2\pi a^3}{3},$$

since $2x\sqrt{4a^2-x^2}$ = difference of squares of ordinates of CD, and BD; also volume generated by AE

$$= \left(\int_{x=a\sqrt{3}} - \int_{x=0}\right) \pi \left(-ax + x\sqrt{4a^2 - x^2}\right) = \frac{5\pi a^3}{6};$$

therefore whole volume $= \pi a^3 \left(\frac{23}{6} + \frac{2}{3} + \frac{5}{6}\right) = \frac{16\pi a^3}{3}$.

Ex. 2. If a circle whose radius $= a$, revolve about an axis in its plane, and c = length of the perpendicular dropped from its center on the axis, the volume and surface of the figure generated are respectively $2\pi^2a^2c$, $4\pi^2ac$.

141. We shall now give a few instances of finding the volumes of figures generated by the motion of a plane surface

parallel to itself, and increasing or decreasing in magnitude after a given law; it is clear a figure of revolution is only a particular case of this, the generating plane surface being a circle whose radius varies so as always to be equal to the ordinate of the curve or directrix. In other cases, if u represent the area of the generating plane surface in terms of x, and a the angle which it makes with the axis of x, we shall have $d_x V = u \sin a$, and $\therefore\ V = \sin a \int_x u$.

Ex. 1. To find the volume of a pyramid.

Let abc (Fig. 19.) be a section of the pyramid made by a plane parallel to the base ABC; then the pyramid may be supposed to be generated by the motion of the triangle abc parallel to itself, and increasing in magnitude so as always to have its angular points in the lines VA, VB, VC. The triangles abc, ABC are manifestly equiangular,

$$\therefore\ \frac{\text{area } abc}{\text{area } ABC} = \frac{(bc)^2}{(BC)^2} = \frac{(Vc)^2}{(VC)^2} = \frac{(Vn)^2}{(VN)^2},$$

VN being a perpendicular from the vertex on the plane of the base;

$$\text{or } u = \frac{Ax^2}{h^2}, \text{ if } VN = h,\ Vn = x, \text{ area } ABC = A;$$

therefore volume of pyramid

$$= \left(\int_{x=h} - \int_{x=0}\right) \frac{Ax^2}{h^2} = \frac{Ah}{3} = \tfrac{1}{3}\,(\text{prism of same base and altitude}).$$

This result is true of any pyramid; or of an oblique cone upon any base.

Ex. 2. To find the volume of any segment of a paraboloid.

Let QAq (Fig. 8.) be a segment of a paraboloid whose axis is AD, then the bounding plane surface QGq is an ellipse; suppose it perpendicular to the plane of the paper and to meet a section of the paraboloid through its axis made by that plane in Qq. Bisect Qq in V, and draw PV parallel to AD, and let PVG be a section of the paraboloid made by a plane through PV perpendicular to the plane of the paper; then PgG is a parabola similar to the generating one, and QV and VG (which is perpendicular to QV) are the semi-axes of QGq; and any parallel section Mgm is an ellipse with semi-axes MN, Ng, by the motion of which parallel to itself the paraboloid may be supposed to be generated.

Let $PN = x$, $SP = p$; then $MN = 2\sqrt{px}$, $Ng = 2\sqrt{mx}$,

$\therefore$ area $Mgm = 4\pi x\sqrt{mp}$; and the inclination of PN to the plane of the moveable area is $\angle PNm = \theta$,

$$\therefore\ d_x V = 4\pi\sqrt{mp}\sin\theta x,$$

$$V = 2\pi\sqrt{mp}\sin\theta x^2,\ \text{(const.} = 0);$$

$$\therefore \text{ volume } QPq = 2\pi\sqrt{mp}\sin\theta.(PV)^2 = \tfrac{1}{2}4\pi\sqrt{mp}.PV.PV\sin\theta$$

$$= \tfrac{1}{2}(\text{circumscribing cylinder } tq).$$

Ex. 3. If about a circle which generates by its motion a solid of revolution, a regular polygon be circumscribed in every position, so that the points of contact may be always in the same planes, to find the volume, and area of the surface, of the figure generated.

Since the area and perimeter of the polygon will be always proportional to the square, and the simple power of the ordinate, if V and S be the volume, and area of the surface generated, we shall have

$$d_x V = my^2,\ d_x S = ny\,d_x s.$$

Suppose the directrix a circle, and the polygon a square, the figure generated is called a groin;

$$\text{and since } m = 4,\ n = 8,\ y^2 = 2ax - x^2,\ d_x s = \frac{a}{y},$$

$$V = 4\int_x(2ax - x^2) = \frac{8a^3}{3} \text{ between the limits } x = 0,\ x = a,$$

$$S = 8\int_x a = 8a^2, \text{ between the same limits.}$$

142. We can always approximate to the area of a curve by the method of equidistant ordinates.

Let it be required to find the area *PNKU* (Fig. 20.) included between the ordinates *PN*, *KU*; divide their distance *NK* into an odd number $(n-1)$ of parts, each $= h$, and call the ordinates drawn through the points of division $a_2, a_3, \ldots a_{n-1}$, and the extreme ordinates a_1, a_n. Through the extremities of the three first describe a parabola *PQR* having its axis parallel to the ordinates of the curve; this is possible, because a parabola can be made to satisfy four conditions; draw the chord *PR* which will be bisected in *V* by the intermediate ordinate, and will be parallel to the tangent of the parabola at *Q*; therefore (Ex. 8. Art. 137.) area of parabolic segment $PQRV = \frac{2}{3}$ circumscribing parallelogram

$$= \frac{2}{3} QV \cdot NL = \frac{4h}{3} \left\{a_2 - \tfrac{1}{2}(a_1 + a_3)\right\},$$

and area of trapezium $PRLN = 2h \cdot \dfrac{a_1 + a_3}{2}$,

therefore whole area

$$PQRLN = h\left\{a_1 + a_3 + \frac{4}{3}a_2 - \frac{2}{3}(a_1 + a_3)\right\} = \frac{h}{3}(a_1 + 4a_2 + a_3);$$

similarly, describing a parabola through the points *R*, *S*, *T*,

$$\text{area } RSTML = \frac{h}{3}(a_3 + 4a_4 + a_5), \text{ \&c.} = \text{\&c.}$$

Hence, by addition, the sum of the parabolic areas, which is a near approximation to the curvilinear area *PNKU*,

$$= \frac{h}{3}(a_1 + a_n + 4a_2 + 2a_3 + 4a_4 + 2a_5 + \text{\&c.} + 4a_{n-1}).$$

Hence the Rule. Add the first and last ordinates to four times the sum of the even, and twice the sum of the odd ones, and multiply by $\frac{1}{3}$ the common distance of the ordinates.

SECTION IX.

ELLIPTIC FUNCTIONS.

ART. 143. THE last case of integrating explicit functions of the variable which we shall consider, is an extensive class of transcendents, of which the elementary forms are

$$F_c(\theta) = \int_\theta \frac{1}{\sqrt{1 - c^2 (\sin\theta)^2}}, \quad E_c(\theta) = \int_\theta \sqrt{1 - c^2 (\sin\theta)^2},$$

$$\text{and } \Pi_c(n, \theta) = \int_\theta \frac{1}{\{1 + n(\sin\theta)^2\} \sqrt{1 - c^2 (\sin\theta)^2}}.$$

These three, denoted by the symbols F, E, Π, are called respectively Elliptic Functions of the first, second, and third orders; they are all comprized in

$$\int_\theta \frac{a + b(\sin\theta)^2}{1 + n(\sin\theta)^2} \cdot \frac{1}{\sqrt{1 - c^2 (\sin\theta)^2}},$$

which is the general form of elliptic functions. The origin of each of the integrals is $\theta = 0$, and its extent is determined by the variable θ which is called its amplitude; c is always < 1, and is called the modulus, and $\sqrt{1 - c^2}$, denoted by b, is called the complement of the modulus; the constant n which is found in $\Pi_c(n, \theta)$ is called its parameter, and may be positive or negative. The reason why they are called elliptic functions is, that $aE_c(\theta)$ is the expression for the length of an elliptic arc found in Ex. 11. Art. 137; and $F_c(\theta)$ can be represented by the arcs of two ellipses, as we shall shew; the appellation is less appropriate as far as regards elliptic functions of the third order. When the superior limit of the integrals is $\theta = \frac{\pi}{2}$, they are called *complete* functions, and are denoted F_c, E_c, $\Pi_c(n)$, suppressing the amplitude. The quantity $\sqrt{1 - c^2 (\sin\theta)^2}$ is denoted by $\Delta_c(\theta)$, or by Δ simply in investigations in which only one modulus and one amplitude enter.

144. In the expressions $F_c(\theta)$, $E_c(\theta)$, the modulus c answers to the base of a system of logarithms; and tables to a certain extent have been constructed by Legendre, whereby their numerical values, corresponding to a given modulus and amplitude, may be found. Since $\Pi_c(n, \theta)$ involves two constants, the formation of tables of elliptic functions of the third order is almost impracticable; but there are many cases where they can be expressed by functions of the first and second orders; when they are complete, this can always be effected.

145. All integrals comprised under the form

$$\int_x \frac{R(x)}{\sqrt{a + bx + cx^2 + ex^3 + fx^4}}$$

can be made to depend upon the elementary integrals described above; we perceive, therefore, the importance of this theory, and the extension it gives to the domains of Analysis. The object of the present Chapter is first, to reduce $\int_x \frac{R(x)}{\sqrt{X}}$ to the simplest integrals of its class, and to shew that they are identical with the three elliptic functions; and secondly, to give the mode of calculating the numerical values of the latter, in order that Problems dependent upon integrals of the form $\int_x \frac{R(x)}{\sqrt{X}}$ may receive the same complete development, as if they depended upon integrals capable of being expressed by logarithms or trigonometrical functions.

146. The analysis of $\int_x \frac{R(x)}{\sqrt{a+bx+cx^2+ex^3+fx^4}}$ is most conveniently effected by three principal steps.

1. Transformation into an integral of the form

$$\int_z \frac{R(z^2)}{\sqrt{(f+gz^2)(h+kz^2)}}.$$

2. Transformation of the above into $\int_\theta \frac{R\{(\sin\theta)^2\}}{\sqrt{1-c^2(\sin\theta)^2}}$.

3. Reduction of the above to

$$\int_\theta \left(\frac{A}{1+n(\sin\theta)^2}+B+C(\sin\theta)^2\right)\frac{1}{\sqrt{1-c^2(\sin\theta)^2}}.$$

These will form the subjects of the following Articles.

147. To transform

$$\int_x \frac{R(x)}{\sqrt{a+bx+cx^2+ex^3+fx^4}} \quad \text{into} \quad \int_z \frac{R(z^2)}{\sqrt{(f+gz^2)(h+kz^2)}}.$$

We shall begin with the transformation of $\int_x \frac{1}{\sqrt{X}}$, from which the other can be easily deduced.

Let X be decomposed into two real quadratic factors, so that we may have

$$a+bx+cx^2+ex^3+fx^4 = (\alpha+2\beta x+\gamma x^2)(\lambda+2\mu x+\nu x^2);$$

this, we know, is always possible, the determination of the coefficients α, β, &c. depending upon a cubic equation whose last term is negative, and which, therefore, has a real positive root; also let $x=\frac{p+qz}{1+z}$, p and q being two unknown constants;

$$\therefore X = \left\{\alpha(1+z)^2 + 2\beta(1+z)(p+qz) + \gamma(p+qz)^2\right\} \times$$

$$\left\{\lambda(1+z)^2 + 2\mu(1+z)(p+qz) + \nu(p+qz)^2\right\} \cdot \frac{1}{(1+z)^4};$$

and in order that the odd powers of z may vanish, we must have

$$\alpha + \beta(p+q) + \gamma pq = 0, \quad \lambda + \mu(p+q) + \nu pq = 0.$$

These equations of course give possible values for pq and $p+q$; but the values of p and q will not be possible, unless $(p+q)^2 - 4pq$ be positive; that this is the case, may be thus shewn.

1. Let the roots of $X = 0$ be not all real, and let $\alpha + 2\beta x + \gamma x^2 = 0$ have its roots imaginary so that $\alpha\gamma > \beta^2$; now the former of the above equations gives

$$(p+q)^2 = \left(\frac{\alpha + \gamma pq}{\beta}\right)^2,$$

$$\therefore (p-q)^2 = \left(\frac{\alpha+\gamma pq}{\beta}\right)^2 - 4pq = \left(\frac{\alpha+\gamma pq}{\beta} - \frac{2\beta}{\gamma}\right)^2 + \frac{4\alpha\gamma - 4\beta^2}{\gamma^2},$$

which is positive, whether the roots of the other quadratic factor be real or not. In the same manner, if it be the factor $\lambda + 2\mu x + \nu x^2 = 0$ whose roots are imaginary, we may shew that the value of $(p-q)^2$ is positive.

2. Let the roots of $X = 0$ be real, and represented by a, b, c, d in order of magnitude; and among the three ways in which X can be resolved into its quadratic factors, let that be taken which groups the two greatest, and two least roots together, so that

$$\alpha + 2\beta x + \gamma x^2 = \gamma(x-a)(x-b),$$

$$\lambda + 2\mu x + \nu x^2 = \nu(x-c)(x-d);$$

then the equations for determining p and q are

$$ab - \tfrac{1}{2}(a + b)(p + q) + pq = 0,$$

$$cd - \tfrac{1}{2}(c + d)(p + q) + pq = 0;$$

from which, by forming the value of $(p + q)^2 - 4pq$, and reducing, we shall obtain

$$\frac{(p - q)^2}{2} = \frac{(a - c)(a - d)(b - c)(b - d)}{(a + b - c - d)^2},$$

which is positive, $\because$ a and b are both greater than c and d.

Hence X can be reduced to the form

$$(f + gz^2)(h + kz^2) \cdot \frac{1}{(1 + z)^4};$$

$$\text{also } d_z x = -\frac{(p - q)}{(1 + z)^2};$$

therefore the proposed integral $\int_z \frac{d_z x}{\sqrt{X}}$ will be transformed into

$$A \int_z \frac{1}{\sqrt{(f + gz^2)(h + kz^2)}}.$$

It is not necessary that X should have all its terms, we may have $f = 0$, or $e = 0$; the only restriction is, that we must not have b and e each $= 0$, and $4af > c^2$; for then the quadratic factors of X are imaginary.

148. By the substitution of the preceding Art., $R(x)$ becomes a rational function of z, which, whether it be integral or fractional, may be reduced to the form

$$R_1(z^2) + zR_2(z^2); \quad \text{hence} \int_x \frac{R(x)}{\sqrt{X}} \text{ becomes}$$

$$A \int_z \frac{R_1(z^2)}{\sqrt{(f + gz^2)(h + kz^2)}} + A \int_z \frac{z R_2(z^2)}{\sqrt{(f + gz^2)(h + kz^2)}},$$

the latter of which, by making $x^2 = y$, may be further reduced to

$$\frac{A}{2}\int_y \frac{R(y)}{\sqrt{(f+gy)(h+ky)}},$$

and is integrable by the common rules, Art. 49, Case III.

Hence $\int_x \frac{R(x)}{\sqrt{X}}$ may be made to depend upon

$$\int_x \frac{R_1(x^2)}{\sqrt{(f+gx^2)(h+kx^2)}}.$$

149. To transform

$$\int_x \frac{R(x^2)}{\sqrt{(f+gx^2)(h+kx^2)}} \text{ into } \int_\theta \frac{R\{(\sin\theta)^2\}}{\sqrt{1-c^2(\sin\theta)^2}},\ c<1.$$

We shall begin with the transformation of

$$\int_x \frac{1}{\sqrt{(f+gx^2)(h+kx^2)}} \text{ or } \int_x \frac{1}{\sqrt{Z}}.$$

This is most conveniently done by considering separately the different forms which each of the factors of

$$\sqrt{(f+gx^2)(h+kx^2)}$$

may have, and the different ways in which they may be combined; it will be seen that every possible case is comprized in the following.

1. Let $Z = m^2(1+p^2x^2)(1-q^2x^2)$.

Since qx cannot exceed 1, let $qx = \cos\theta$,

$$\therefore \int_x \frac{1}{\sqrt{Z}} = \int_\theta \frac{-\frac{1}{q}\sin\theta}{\sqrt{m^2(\sin\theta)^2\left\{1+\frac{p^2}{q^2}(\cos\theta)^2\right\}}}$$

$$= -\frac{1}{mq}\int_\theta \frac{1}{\sqrt{1+\frac{p^2}{q^2}-\frac{p^2}{q^2}(\sin\theta)^2}} = -\frac{c}{mp}\int_\theta \frac{1}{\sqrt{1-c^2(\sin\theta)^2}},$$

$$\text{where } c = \frac{p}{\sqrt{p^2+q^2}}.$$

2. Let $Z = m^2(1+p^2x^2)(x^2-q^2)$, and since x is always $> q$, let $x = q\sec\theta$,

$$\therefore \int_x \frac{1}{\sqrt{Z}} = \int_\theta \frac{q\sec\theta\tan\theta}{mq\tan\theta\sqrt{1+(pq\sec\theta)^2}}$$

$$= \frac{1}{m}\int_\theta \frac{1}{\sqrt{1+(pq)^2-(\sin\theta)^2}} = \frac{c}{m}\int_\theta \frac{1}{\sqrt{1-c^2(\sin\theta)^2}},$$

$$\text{where } c = \frac{1}{\sqrt{1+p^2q^2}}.$$

3. Let $Z = m^2(1+p^2x^2)(1+q^2x^2)$, and suppose $p > q$; let $px = \tan\theta$,

$$\therefore \int_x \frac{1}{\sqrt{Z}} = \int_\theta \frac{\frac{1}{p}(\sec\theta)^2}{m\sec\theta\sqrt{1+\frac{q^2}{p^2}(\tan\theta)^2}}$$

$$= \frac{1}{mp}\int_\theta \frac{1}{\sqrt{(\cos\theta)^2+\frac{q^2}{p^2}(\sin\theta)^2}} = \frac{1}{mp}\int_\theta \frac{1}{\sqrt{1-c^2(\sin\theta)^2}},$$

$$\text{where } c = \frac{\sqrt{p^2-q^2}}{p}.$$

4. Let $Z = m^2(1 - p^2x^2)(1 - q^2x^2)$, and suppose $p > q$; since px cannot exceed 1, let $px = \sin\theta$,

$$\therefore \int_z \frac{1}{\sqrt{Z}} = \int_\theta \frac{\frac{1}{p}\cos\theta}{m\cos\theta\sqrt{1 - \frac{q^2}{p^2}(\sin\theta)^2}}$$

$$= \frac{1}{mp}\int_\theta \frac{1}{\sqrt{1 - c^2(\sin\theta)^2}}, \text{ where } c = \frac{q}{p}.$$

5. Let $Z = m^2(x^2 - q^2)(p^2 - x^2)$, and suppose $p > q$; let $x = q\sec\theta$,

$$\therefore \int_z \frac{1}{\sqrt{Z}} = \int_\theta \frac{q\sec\theta\tan\theta}{mq\tan\theta\sqrt{p^2 - q^2(\sec\theta)^2}}$$

$$= \frac{1}{m}\int_\theta \frac{1}{\sqrt{p^2 - q^2 - p^2(\sin\theta)^2}} = \frac{1}{mp}\int_\theta \frac{1}{\sqrt{c^2 - (\sin\theta)^2}},$$

where $c = \frac{\sqrt{p^2 - q^2}}{p}$. Let $\sin\theta = c\sin\phi$,

$$\therefore \int_z \frac{1}{\sqrt{Z}} = \frac{1}{mp}\int_\phi \frac{c\cos\phi}{\cos\theta \,.\, c\cos\phi} = \frac{1}{mp}\int_\phi \frac{1}{\sqrt{1 - c^2(\sin\phi)^2}}.$$

This transformation might have been effected immediately, by making

$$x^2 = \frac{q^2}{1 - (\sin\theta)^2} = \frac{q^2}{1 - c^2(\sin\phi)^2}.$$

150. All the above substitutions are comprehended in the form

$$x^2 = \frac{m_, + n_,(\sin\theta)^2}{p_, + q_,(\sin\theta)^2};$$

if then we make this substitution in

$$\int_z \frac{R_1(z^2)}{\sqrt{(f+gz^2)(h+kz^2)}},$$

it will be transformed into

$$\int_\theta \frac{R\{(\sin\theta)^2\}}{\sqrt{1-c^2(\sin\theta)^2}}.$$

151. When the quadratic factors of X are imaginary, as noticed at the end of Art. 147., it will be of the form

$$X = \lambda^2 + 2\lambda\mu x^2 \cos a + \mu^2 x^4.$$

$$\text{Make } x = \sqrt{\frac{\lambda}{\mu}} \tan\frac{\theta}{2},$$

$$\therefore\ \lambda^2 + 2\lambda\mu x^2\cos a + \mu^2 x^4 = \lambda^2\left\{1 + 2\cos a\left(\tan\frac{\theta}{2}\right)^2 + \left(\tan\frac{\theta}{2}\right)^4\right\}$$

$$= \lambda^2\left(\sec\frac{\theta}{2}\right)^4\left\{1 - \left(\sin\frac{a}{2}\right)^2(\sin\theta)^2\right\},$$

$$\therefore\ \int_x \frac{1}{\sqrt{X}} = \int_\theta \frac{d_\theta x}{\lambda\left(\sec\frac{\theta}{2}\right)^2\sqrt{1-\left(\sin\frac{a}{2}\right)^2(\sin\theta)^2}}$$

$$= \frac{1}{2\sqrt{\lambda\mu}}\int_\theta \frac{1}{\sqrt{1-c^2(\sin\theta)^2}}, \text{ where } c = \sin\frac{a}{2}.$$

152. To investigate formulæ of reduction for

$$\int_\theta \frac{R\{(\sin\theta)^2\}}{\sqrt{1-c^2(\sin\theta)^2}} \text{ or } \int_x \frac{R(x^2)}{\sqrt{1-(1+c^2)x^2+c^2x^4}} \text{ (making } \sin\theta = x).$$

First let $R(x^2)$ be an integral function, then each term in $\int_x \frac{R(x^2)}{\sqrt{X}}$ will be of the form $\int_x \frac{x^{2m}}{\sqrt{X}}$, the formula of reduction for which is (Art. 58. Ex. 1.)

$$\int_x \frac{x^{2m}}{\sqrt{X}} = \frac{x^{2m-3}\sqrt{X}}{(2m-1)c^2} + \frac{2m-2}{2m-1} \cdot \frac{1+c^2}{c^2} \int_x \frac{x^{2m-2}}{\sqrt{X}}$$

$$- \frac{2m-3}{2m-1} \cdot \frac{1}{c^2} \int_x \frac{x^{2m-4}}{\sqrt{X}};$$

or, restoring the value of x, and putting Δ for $\sqrt{1-c^2(\sin\theta)^2}$,

$$\int_\theta \frac{(\sin\theta)^{2m}}{\Delta} = \frac{\cos\theta\,(\sin\theta)^{2m-3}\Delta}{(2m-1)\,c^2} + \frac{2m-2}{2m-1}\,\frac{1+c^2}{c^2} \int_\theta \frac{(\sin\theta)^{2m-2}}{\Delta}$$

$$- \frac{2m-3}{2m-1}\,\frac{1}{c^2} \int_\theta \frac{(\sin\theta)^{2m-4}}{\Delta}.$$

By this formula, $\int_\theta \frac{(\sin\theta)^{2m}}{\Delta}$ can be made to depend upon

$$\int_\theta \frac{(\sin\theta)^2}{\Delta} \text{ and } \int_\theta \frac{1}{\Delta},$$

that is, upon an integral of the form

$$\int_\theta \{A + B(\sin\theta)^2\} \frac{1}{\Delta}.$$

153. Secondly, let $R(x^2)$ be a fraction; it may be resolved into partial fractions of the general form $\frac{N}{(1+nx^2)^m}$, their number being equal to the dimension in x^2 of its denominator, and N and n being constants real or imaginary. Hence every term of $\int_x \frac{R(x^2)}{\sqrt{X}}$ will be of the form

$$N \int_x \frac{1}{(1+nx^2)^m \sqrt{X}}.$$

Make $1 + nx^2 = z$,

$$\therefore\ X = 1 - \frac{1+c^2}{n}(z-1) + \frac{c^2}{n^2}(z-1)^2$$

$$= 1 + \frac{1+c^2}{n} + \frac{c^2}{n^2} - z\left(\frac{1+c^2}{n} + \frac{2c^2}{n^2}\right) + z^2\frac{c^4}{n^2}$$

$= \alpha + \beta z + \gamma z^2$, suppose; also, $d_z x = \dfrac{1}{2\sqrt{n}}\,\dfrac{1}{\sqrt{z-1}}$;

$$\therefore\ N\int_x \frac{1}{(1+nx^2)^m\sqrt{X}} \text{ becomes}$$

$$\frac{N}{2\sqrt{n}}\int_z \frac{1}{z^m\sqrt{(z-1)(\alpha+\beta z+\gamma z^2)}} \text{ or } \frac{N}{2\sqrt{n}}\int_z \frac{1}{z^m\sqrt{Z}},$$

putting $Z = -\alpha + (\alpha-\beta)z + (\beta-\gamma)z^2 + \gamma z^3$.

This integral, by Art. 61, can be made to depend upon three similar ones in which the index of z is increased, by differentiating $z^{-m+1}\sqrt{Z}$, which gives

$$\frac{z^{-m}}{\sqrt{X}}\left\{-(m-1)Z + \tfrac{1}{2}z\,d_z Z\right\}$$

$$= \frac{z^{-m}}{\sqrt{Z}}\left\{-(m-1)(-\alpha+(\alpha-\beta)z+(\beta-\gamma)z^2+\gamma z^3)\right.$$
$$\left. + \tfrac{1}{2}z(\alpha-\beta+2(\beta-\gamma)z+3\gamma z^2)\right\}$$

$$= \frac{z^{-m}}{\sqrt{Z}}\left\{(m-1)\alpha - \left(m-\frac{3}{2}\right)(\alpha-\beta)z - (m-2)\right.$$

$$\left.(\beta-\gamma)z^2 - \left(m-\frac{5}{2}\right)\gamma z^3\right\};$$

therefore, integrating,

$$z^{-m+1}\sqrt{Z} = (m-1)\,\alpha\int_z \frac{1}{z^m\sqrt{Z}} - \left(m-\frac{3}{2}\right)(\alpha-\beta)\int_z \frac{1}{z^{m-1}\sqrt{Z}}$$

$$-(m-2)(\beta-\gamma)\int_z \frac{1}{z^{m-2}\sqrt{Z}} - \left(m-\frac{5}{2}\right)\gamma\int_z \frac{1}{z^{m-3}\sqrt{Z}}.$$

Hence dividing by $2\sqrt{n}$, and restoring the value of z, in which case $\sqrt{Z} = \sqrt{n}\sin\theta\cos\theta\,\Delta$,

and $$\frac{1}{2\sqrt{n}}\int_z \frac{1}{z^m\sqrt{Z}} = \int_\theta \frac{1}{\{1+n(\sin\theta)^2\}^m\,\Delta} = V_m \text{ suppose,}$$

$$\frac{\sin\theta\cos\theta\,\Delta}{\{1+n(\sin\theta)^2\}^{m-1}} = (2m-2)\,\alpha\,V_m - (2m-3)(\alpha-\beta)\,V_{m-1}$$

$$-(2m-4)(\beta-\gamma)\,V_{m-2} - (2m-5)\,\gamma\,V_{m-3}.$$

Hence V_m will at last depend upon V_1, V_0, V_{-1}, that is upon an integral of the form

$$\int_\theta\left\{\frac{N}{1+n(\sin\theta)^2} + A + B(\sin\theta)^2\right\}\frac{1}{\Delta};$$

but $$\int_\theta\{A+B(\sin\theta)^2\}\frac{1}{\Delta} = \int_\theta\left(A+\frac{B}{c^2}-\frac{B}{c^2}\Delta^2\right)\frac{1}{\Delta}$$

$$= \left(A+\frac{B}{c^2}\right)\int_\theta\frac{1}{\Delta} - \frac{B}{c^2}\int_\theta\Delta,$$

hence whether $R\{(\sin\theta)^2\}$ be integral or fractional, it is demonstrated that every term in $\int_\theta\frac{R\{(\sin\theta)^2\}}{\Delta}$ can be made to depend upon one or more of the elementary forms

$$F_c(\theta),\quad E_c(\theta),\quad \Pi_c(n,\theta).$$

154. We shall now apply the preceding methods to obtain certain results, some of which will be required in the advanced parts of the subject.

Ex. 1. To express $\int_x \frac{1}{\sqrt{1-x^3}}$ by an elliptic function.

Assuming $x = \frac{p+qz}{1+z}$, we find $p = 1 + \sqrt{3}$, $q = 1 - \sqrt{3}$; and if $m^2 = \frac{2-\sqrt{3}}{2+\sqrt{3}}$, or $m = \tan 15^0$, the transformed integral is $-\frac{\sqrt{1+m^2}}{\sqrt[4]{3}} \int_z \frac{1}{\sqrt{(z^2-1)(1+m^2z^2)}}$; hence, making $z = \sec\theta$,

$$\int_x \frac{1}{\sqrt{1-x^3}} = -\frac{1}{\sqrt[4]{3}} \int_\theta \frac{1}{\sqrt{1-c^2(\sin\theta)^2}} = -\frac{1}{\sqrt[4]{3}} F_c(\theta) + \text{const.}$$

where $c = \cos 15^0 = \frac{1+\sqrt{3}}{2\sqrt{2}}$, and $x = -\sqrt{3}\left(\tan\frac{\theta}{2}\right)^2 + 1$.

Ex. 2. To express $\int_\theta \frac{1}{\{1-c^2(\sin\theta)^2\}^{\frac{3}{2}}}$ by an elliptic function.

The formula of reduction for $\int_\theta \frac{1}{\{1-c^2(\sin\theta)^2\}^m\Delta} = V_m$ is

$$\frac{c^2 \sin\theta\cos\theta}{\Delta^{2m-3}} = -(2m-3)\,b^2 V_{m-1} + (2m-4)(1+b^2)\,V_{m-2}$$
$$-(2m-5)\,V_{m-3},$$

making $n = -c^2$, in the general formula, so that $a = 0$;

$$\text{let } m = 2, \quad \therefore \frac{c^2\sin\theta\cos\theta}{\Delta} = -b^2 V_1 + V_{-1},$$

$$\therefore\ V_1 = \frac{1}{b^2} V_{-1} - \frac{c^2}{b^2} \frac{\sin\theta\cos\theta}{\Delta};$$

$$\text{or } \int_\theta \frac{1}{\Delta^3} = \frac{1}{b^2} \int_\theta \Delta - \frac{c^2}{b^2} \frac{\sin\theta\cos\theta}{\Delta} = \frac{1}{b^2} E_c(\theta) - \frac{c^2}{b^2} \frac{\sin\theta\cos\theta}{\Delta}.$$

By making $\cot\theta = b\tan\phi$, we find

$$\int_\theta \frac{1}{\Delta^{2m+1}} = -\frac{1}{b^{2m}} \int_\phi \{1 - c^2(\sin\phi)^2\}^{\frac{2m-1}{2}};$$

$$\text{hence } \left(\int_{\theta=\frac{\pi}{2}} - \int_{\theta=0}\right) \frac{1}{\Delta^{2m+1}} = \frac{1}{b^{2m}} \left(\int_{\theta=\frac{\pi}{2}} - \int_{\theta=0}\right) \Delta^{2m-1}.$$

Ex. 3. To express $\int_\theta \frac{1}{\{1 + n(\sin\theta)^2\}^2\Delta}$ by elliptic functions.

Restoring, in the general formula, the values of α, β, γ, and making $m = 2$, we find

$$\frac{\sin\theta\cos\theta\,\Delta}{1 + n(\sin\theta)^2} = 2\left(1 + \frac{1 + c^2}{n} + \frac{c^2}{n^2}\right) V_2$$

$$- \left\{1 + \frac{2}{n}(1 + c^2) + \frac{3c^2}{n^2}\right\} V_1 + \frac{c^2}{n^2} V_{-1};$$

$$\text{but } \frac{c^2}{n^2} V_{-1} = \frac{c^2}{n^2} \int_\theta \frac{1 + n(\sin\theta)^2}{\Delta} = \frac{1}{n} \int_\theta \left(\frac{c^2}{n} + 1 - \Delta^2\right) \frac{1}{\Delta}$$

$$= \frac{c^2}{n^2} F_c(\theta) - \frac{1}{n}\{E_c(\theta) - F_c(\theta)\},$$

$$\therefore\ 2\left(1 + \frac{1}{n}\right)\left(1 + \frac{c^2}{n}\right) \int_\theta \frac{1}{\{1 + n(\sin\theta)^2\}^2\Delta}$$

$$= \frac{\sin\theta\cos\theta\Delta}{1+n(\sin\theta)^2} + \left\{1 + \frac{2}{n}(1+c^2) + \frac{3c^2}{n^2}\right\}\Pi_c(n,\theta)$$

$$-\frac{c^2}{n^2}F_c(\theta) + \frac{1}{n}\{E_c(\theta) - F_c(\theta)\}.$$

We shall next give the particular reduction of the following algebraic expression to the elementary forms.

155. To reduce $u = \int_x\left(\frac{f+gx^2}{\sqrt{a^2+2a\beta\cos a x^2+\beta^2x^4}} - \frac{g}{\beta}\right)$

$$\text{or} = \int_x\left(\frac{f+gx^2}{\sqrt{X}} - \frac{g}{\beta}\right),$$

to functions of the first and second orders.

The term $-\frac{g}{\beta}$ is added, in order that the value of the integral may continue finite, when x is indefinitely increased In the first place, in order to make the quadratic factors of the quantity under the radical real,

$$\text{let } \beta x^2 + a\cos a + \sqrt{a^2+2a\beta\cos a x^2+\beta^2x^4} = 2az^2,$$

$$\therefore\ x^2 = \frac{a}{\beta}\left(z^2 - \cos a - \frac{(\sin a)^2}{4z^2}\right),$$

$$\text{and } \sqrt{X} = 2az^2 - (\beta x^2 + a\cos a) = a\left(z^2 + \frac{(\sin a)^2}{4z^2}\right);$$

$$\therefore\ xd_zx = \frac{a}{\beta}\left(z + \frac{(\sin a)^2}{4z^3}\right) = \frac{\sqrt{X}}{\beta z},$$

$$\therefore\ d_zu = \left(\frac{f+gx^2}{\sqrt{X}} - \frac{g}{\beta}\right)\frac{\sqrt{X}}{\beta xz} = \left(f + gx^2 - \frac{g}{\beta}\sqrt{X}\right)\frac{1}{\beta xz};$$

$$\text{but } x^2 - \frac{\sqrt{X}}{\beta} = -\frac{a}{\beta}\cos a - \frac{a}{\beta}\frac{(\sin a)^2}{2z^2},$$

$$\therefore\ d_z u = \frac{f - \frac{ga}{\beta}\left\{\cos a + \frac{1}{2z^2}(\sin a)^2\right\}}{\sqrt{a\beta}\sqrt{z^4 - z^2\cos a - \frac{1}{4}(\sin a)^2}},$$

where the quantity under the radical

$$= \left\{z^2 - \left(\cos\frac{a}{2}\right)^2\right\}.\left\{z^2 + \left(\sin\frac{a}{2}\right)^2\right\}.$$

Now make $z = \cos\frac{a}{2}\sec\theta$, $\therefore\ d_\theta z = \cos\frac{a}{2}\sec\theta\tan\theta$,

$$\text{and } \left\{z^2 - \left(\cos\frac{a}{2}\right)^2\right\}.\left\{z^2 + \left(\sin\frac{a}{2}\right)^2\right\}$$

$$= \left(\cos\frac{a}{2}\right)^2(\tan\theta)^2\left\{\left(\cos\frac{a}{2}\sec\theta\right)^2 + \left(\sin\frac{a}{2}\right)^2\right\}$$

$$= \left(\cos\frac{a}{2}\tan\theta\sec\theta\right)^2.\left\{1 - \left(\sin\frac{a}{2}\right)^2(\sin\theta)^2\right\};$$

$$\therefore\ d_\theta u = \frac{f - g\frac{a}{\beta}\left\{\cos a + 2\left(\sin\frac{a}{2}\right)^2(\cos\theta)^2\right\}}{\sqrt{a\beta}\sqrt{1 - \left(\sin\frac{a}{2}\right)^2(\sin\theta)^2}}$$

$$= \frac{f\beta - ga + 2gac^2(\sin\theta)^2}{\beta\sqrt{a\beta}\sqrt{1 - c^2(\sin\theta)^2}},\quad \text{where } c = \sin\frac{a}{2};$$

$$\therefore\ u = \frac{f\beta + ga}{\beta\sqrt{a\beta}}F_c(\theta) - \frac{2ga}{\beta\sqrt{a\beta}}E_c(\theta),$$

the origin of the integral being $\theta = 0$, which corresponds to $x = 0$, because

$$x = \left(\frac{a}{\beta}\right)^{\frac{1}{2}}\tan\theta\sqrt{1 - c^2(\sin\theta)^2};\ \text{hence when } \theta = \frac{\pi}{2},\ x = \infty,$$

and to obtain the value of u between the limits $x = 0$, $x = \infty$, we must take complete functions in the above expression.

Hence, $\int_x \frac{f+gx^2}{\sqrt{X}} = u + \frac{gx}{\beta} = u + \frac{g}{\beta}\sqrt{\frac{a}{\beta}}\tan\theta\sqrt{1-c^2(\sin\theta)^2}.$

156. Having thus shewn the mode of reducing the integral $\int_x \frac{R(x)}{\sqrt{X}}$ to the simplest form it is susceptible of, without diminishing its generality, we proceed to the particular consideration of the elementary forms $F_c(\phi)$ and $E_c(\phi)$; and first to the comparison of functions of the same order relative to the same modulus, which will furnish us with formulæ for their addition and subtraction. Since all the functions involved have the same modulus, it may be suppressed, for the sake of simplifying the notation.

157. If ϕ and ψ are dependent upon one another by the transcendental equation

$$F(\phi) + F(\psi) = F(\sigma),$$

they are also dependent upon one another by the algebraic equation

$$\cos\phi\cos\psi - \sin\phi\sin\psi\sqrt{1-c^2(\sin\sigma)^2} = \cos\sigma.$$

Consider ϕ and ψ each as a function of another variable t; therefore, differentiating the above equation, since σ is a constant,

$$\therefore \frac{d_t\phi}{\sqrt{1-c^2(\sin\phi)^2}} + \frac{d_t\psi}{\sqrt{1-c^2(\sin\psi)^2}} = 0.$$

As we are at liberty to assume ϕ to be any function of t, and then this equation will determine the corresponding function which expresses ψ, let the function of t which expresses ϕ be determined by the equation

$$d_t\phi = \sqrt{1-c^2(\sin\phi)^2}\,; \quad \therefore\ d_t\psi = -\sqrt{1-c^2(\sin\psi)^2}.$$

Square these equations, and differentiate,

$$\therefore\ d_t^2\phi = -\tfrac{1}{2}c^2\sin 2\phi, \quad d_t^2\psi = -\tfrac{1}{2}c^2\sin 2\psi,$$

$$\therefore\ d_t^2(\phi \pm \psi) = -\tfrac{1}{2}c^2(\sin 2\phi \pm \sin 2\psi);$$

$$\text{or, if } \phi + \psi = p, \quad \phi - \psi = q,$$

$$d_t^2 p = -c^2 \sin p \cos q,\ d_t^2 q = -c^2 \cos p \sin q;$$

also $d_t p . d_t q = (d_t \phi)^2 - (d_t \psi)^2 = -c^2 \{(\sin \phi)^2 - (\sin \psi)^2\} = -c^2 \sin p \sin q$;

$$\therefore \frac{d_t^2 p}{d_t p} = d_t q \frac{\cos q}{\sin q} = \frac{d_t (\sin q)}{\sin q},\quad \frac{d_t^2 q}{d_t q} = \frac{d_t (\sin p)}{\sin p},$$

$$\therefore d_t p = C \sin q,\ d_t q = C_1 \sin p \quad (1);$$

and $C_1 \sin p\, d_t p = C \sin q\, d_t q$, or $C_1 \cos p = C \cos q + C_2$.

But if $\psi = 0$, $F(\phi) = F(\sigma)$, $\therefore \phi = \sigma$, and $p = q = \sigma$,

$$\therefore C_2 = (C_1 - C) \cos \sigma;$$

$$\therefore C_1 \cos (\phi + \psi) = C \cos (\phi - \psi) + (C_1 - C) \cos \sigma,$$

or $(C_1 - C) \cos \phi \cos \psi - (C_1 + C) \sin \phi \sin \psi = (C_1 - C) \cos \sigma$.

But making $\psi = 0$ in equations (1), after having put for $d_t p$, $d_t q$, their values

$$\sqrt{1 - c^2 (\sin \phi)^2} - \sqrt{1 - c^2 (\sin \psi)^2},$$

and $\sqrt{1 - c^2 (\sin \phi)^2} + \sqrt{1 - c^2 (\sin \psi)^2}$, we have

$$\sqrt{1 - c^2 (\sin \sigma)^2} - 1 = C \sin \sigma,\ \sqrt{1 - c^2 (\sin \sigma)^2} + 1 = C_1 \sin \sigma,$$

$$\therefore 2\sqrt{1 - c^2 (\sin \sigma)^2} = (C_1 + C) \sin \sigma,\ 2 = (C_1 - C) \sin \sigma;$$

hence, by substitution,

$$\cos \phi \cos \psi - \sin \phi \sin \psi \sqrt{1 - c^2 (\sin \sigma)^2} = \cos \sigma,$$

which is the fundamental equation, from which all properties of elliptic functions, where the modulus remains unchanged, can be deduced.

158. The equation

$$\cos \sigma = \cos \phi \cos \psi - \sin \phi \sin \psi \sqrt{1 - c^2 (\sin \sigma)^2}$$

expresses how the amplitude σ of a function, depends upon the amplitudes of two functions of which it is the sum. If we clear it of the radical which it involves, we find

$$(\cos \phi)^2 + (\cos \psi)^2 + (\cos \sigma)^2 - 2 \cos \phi \cos \psi \cos \sigma$$
$$= 1 - c^2 (\sin \phi \sin \psi \sin \sigma)^2;$$

add $(\cos\psi\cos\sigma)^2$ to both members, and transpose,

$$\therefore\ (\cos\phi - \cos\psi\cos\sigma)^2 = 1 - (\cos\psi)^2 - (\cos\sigma)^2$$
$$+ (\cos\psi\cos\sigma)^2 - c^2(\sin\phi\sin\psi\sin\sigma)^2$$
$$= (\sin\psi\sin\sigma)^2 - c^2(\sin\phi\sin\psi\sin\sigma)^2 = (\sin\psi\sin\sigma)^2\{1 - c^2(\sin\phi)^2\};$$
$$\therefore\ \cos\phi = \cos\psi\cos\sigma + \sin\psi\sin\sigma\sqrt{1 - c^2(\sin\phi)^2};$$

(taking the positive sign, as we must have $\psi = \sigma$, when $\phi = 0$); this equation expresses how the amplitude ϕ of a function, depends upon the amplitudes of two other functions of which it is the difference.

Similarly, $\cos\psi = \cos\phi\cos\sigma + \sin\phi\sin\sigma\sqrt{1 - c^2(\sin\psi)^2}$.

159. Hence, having given two functions $F(\phi)$, $F(\psi)$, if we wish to determine a function $F(\sigma)$ equal to their sum, we must have

$$\cos\sigma = \cos\phi\cos\psi - \sin\phi\sin\psi\sqrt{1 - c^2(\sin\sigma)^2}, \quad (1)$$

which gives $\sin\sigma = \dfrac{\sin\phi\cos\psi\,\Delta(\psi) + \sin\psi\cos\phi\,\Delta(\phi)}{1 - c^2(\sin\phi\sin\psi)^2}$;

if we wish to determine a function $F(\delta)$ equal to their difference, we must have

$$\cos\delta = \cos\phi\cos\psi + \sin\phi\sin\psi\sqrt{1 - c^2(\sin\delta)^2},$$

which gives $\sin\delta = \dfrac{\sin\phi\cos\psi\,\Delta(\psi) - \sin\psi\cos\phi\,\Delta(\psi)}{1 - c^2(\sin\phi\sin\psi)^2}$.

Hence, $\sin\sigma\sin\delta = \dfrac{(\sin\phi)^2 - (\sin\psi)^2}{1 - c^2(\sin\phi\sin\psi)^2}$.

If $c = 0$, these values of $\sin\sigma$, $\sin\delta$, coincide with $\sin(\phi+\psi)$, $\sin(\phi-\psi)$, as they ought, for $F_{c=0}(\phi) = \phi$.

160. If $\sigma = \dfrac{\pi}{2}$, $\quad F = F(\phi) + F(\psi)$,

and the equation connecting ϕ and ψ becomes

$$\sqrt{1 - c^2}\tan\phi\tan\psi = 1, \quad \text{or } b\tan\phi\tan\psi = 1.$$

If $\phi = \psi$, $F(\sigma) = 2F(\phi)$,

$$\text{and } \cos\sigma = (\cos\phi)^2 - (\sin\phi)^2\sqrt{1-c^2(\sin\sigma)^2};$$

which may be transformed into

$$\tan\frac{\sigma}{2} = \tan\phi\sqrt{1-c^2(\sin\phi)^2}, \text{ or } (\sin\phi)^2 = \frac{1-\cos\sigma}{1+\sqrt{1-c^2(\sin\sigma)^2}};$$

the first gives the amplitude σ of a function that is double of a proposed function, and the second gives the amplitude ϕ of a function that is half a given function.

161. The amplitudes σ, ϕ, ψ, may be represented by the sides of a spherical triangle, as the form of the fundamental equation would lead us to infer.

Let the sides *AB*, *AC*, *BC* of the spherical triangle *ABC* (Fig. 21.) be denoted by σ, ϕ, ψ; then the opposite angles *C*, *B*, *A*, will be such that

$$\cos C = \frac{\cos\sigma - \cos\phi\cos\psi}{\sin\phi\sin\psi} = -\sqrt{1-c^2(\sin\sigma)^2},$$

$$\cos B = \sqrt{1-c^2(\sin\phi)^2}, \quad \cos A = \sqrt{1-c^2(\sin\psi)^2};$$

$$\therefore\ c = \frac{\sin C}{\sin\sigma} = \frac{\sin B}{\sin\phi} = \frac{\sin A}{\sin\psi}.$$

These equations agree with the known properties of spherical triangles, and shew that if a spherical triangle be constructed with one obtuse, and two acute angles, such that the ratio of the sine of each angle to the sine of the opposite side is equal to the modulus c, then its three sides σ, ϕ, ψ, satisfy the equation $F(\sigma) = F(\phi) + F(\psi)$, σ being the side opposite to the obtuse angle.

162. Hence we can verify the results of the preceding articles. Let the angle *C* and the opposite side σ remain constant, whilst the other two sides vary; and let *ab* be the consecutive position of *AB*; from the point of intersection *O*

as pole, describe arcs of small circles aa, $B\beta$, therefore $a\beta = Ba$; and $AB = ab$, therefore $Aa = b\beta$, and ultimately $-\delta\phi \cos A = \delta\psi \cos b$;

$$\therefore \text{ limit of } \frac{\cos b}{\cos A} \cdot \frac{\delta\psi}{\delta\phi} = -1, \text{ or } \frac{\cos B}{\cos A} d_{\phi}\psi = -1;$$

$$\therefore \frac{1}{\cos B} + \frac{1}{\cos A} d_{\phi}\psi = 0,$$

$$\text{or } \frac{1}{\sqrt{1 - c^2 (\sin\phi)^2}} + \frac{1}{\sqrt{1 - c^2 (\sin\psi)^2}} d_{\phi}\psi = 0;$$

$$\therefore F(\phi) + F(\psi) = C = F(\sigma), \text{ for } \phi = \sigma, \text{ when } \psi = 0$$

and consequently $F(\psi) = 0$.

163. Hence also we can abbreviate our operations, by availing ourselves of the known relations among the sides of a spherical triangle. Thus, having given two functions to find the amplitude of a function that is equal to their sum, instead of solving equation (1) Art. 159. to find σ, we may proceed in this manner. Let ABC Fig. (22) be a spherical triangle constructed so that its three sides satisfy the equation

$$F(\sigma) = F(\phi) + F(\psi);$$

from the obtuse angle draw CD, the arc of a great circle, perpendicular to AB;

$$\therefore \tan AD = \cos A \tan\phi = \tan\phi \sqrt{1 - c^2 (\sin\psi)^2} = \tan\phi \Delta(\psi);$$

$$\text{similarly, } \tan BD = \tan\psi \Delta(\phi),$$

$$\therefore \sigma = AD + BD = \tan^{-1}\{\tan\phi\Delta(\psi)\} + \tan^{-1}\{\tan\psi\Delta(\phi)\}.$$

Again, supposing $\phi > \psi$, make $CE = \psi$, $AE = \delta$; then the sides of the triangle AEC will manifestly satisfy the equation

$$F(\delta) = F(\phi) - F(\psi),$$

$$\text{and } \delta = AD - BD = \tan^{-1}\{\tan\phi\Delta(\psi)\} - \tan^{-1}\{\tan\psi\Delta(\phi)\}.$$

$$\text{Also } (\sin AD)^2 = \frac{(\sin\phi)^2 - c^2(\sin\psi\sin\phi)^2}{1 - c^2(\sin\psi\sin\phi)^2},$$

$$(\sin BD)^2 = \frac{(\sin\psi)^2 - c^2(\sin\phi\sin\psi)^2}{1 - c^2(\sin\phi\sin\psi)^2};$$

$$\therefore\ \sin\sigma\sin\delta = (\sin AD)^2 - (\sin BD)^2 = \frac{(\sin\phi)^2 - (\sin\psi)^2}{1 - c^2(\sin\phi\sin\psi)^2}.$$

164. Again, let ϕ_{n-1}, ϕ_n, ϕ_{n+1}, denote the amplitudes of functions that are respectively $(n-1)$ times, n times, and $(n+1)$ times, a function whose amplitude is ϕ_1,

$$\therefore\ F(\phi_n) = nF(\phi_1);$$

$$\therefore\ F(\phi_{n+1}) = (n+1)F(\phi_1) = F(\phi_n) + F(\phi_1),$$

$$F(\phi_{n-1}) = (n-1)F(\phi_1) = F(\phi_n) - F(\phi_1);$$

hence, by the preceding Art.

$$\phi_{n+1} = \tan^{-1}\{\tan\phi_n\Delta(\phi_1)\} + \tan^{-1}\{\tan\phi_1\Delta(\phi_n)\},$$

$$\phi_{n-1} = \tan^{-1}\{\tan\phi_n\Delta(\phi_1)\} - \tan^{-1}\{\tan\phi_1\Delta(\phi_n)\};$$

consequently, by addition,

$$\tan(\tfrac{1}{2}\phi_{n+1} + \tfrac{1}{2}\phi_{n-1}) = \tan\phi_n\Delta(\phi_1).$$

Let $n = 1, 2$, &c., $\therefore \tan(\tfrac{1}{2}\phi_2) = \tan\phi_1\Delta(\phi_1)$,

$$\tan(\tfrac{1}{2}\phi_3 + \tfrac{1}{2}\phi_1) = \tan\phi_2\Delta(\phi_1), \text{ \&c.}$$

thus by successive substitutions, we can readily compute the amplitude of a function which is a given multiple of another function.

165. To find the amplitude of a function which is a given part of another, is more difficult; for it requires the solution of the equation between $\sin\phi_n$, and $\sin\phi_1$, which would be formed by eliminating ϕ_2, ϕ_3, &c. ϕ_{n-1} between the equations in the last article, $\tan(\tfrac{1}{2}\phi_2) = \tan\phi_1\Delta(\phi_1)$, $\tan(\tfrac{1}{2}\phi_3 + \tfrac{1}{2}\phi_1) = \tan\phi_2\Delta(\phi_1)$, &c., $\tan(\tfrac{1}{2}\phi_n + \tfrac{1}{2}\phi_{n-2}) = \tan\phi_{n-1}\Delta(\phi_1)$; the degree of which increases very rapidly with n. We shall shew the method in the following example.

Ex. To trisect a complete function.

Let $\phi_3 = \frac{1}{2}\pi$, $\therefore \tan(\frac{1}{4}\pi + \frac{1}{2}\phi_1) = \tan\phi_2 \Delta(\phi_1)$.

But $F(\phi_2) + F(\phi_1) = F$, $\therefore b \tan\phi_1 \tan\phi_2 = 1$;

$$\therefore \frac{1+\sin\phi_1}{\cos\phi_1} = \frac{1}{b}\cot\phi_1 \Delta(\phi_1), \text{ or } b\sin\phi_1 = (1-\sin\phi_1)\Delta(\phi_1),$$

or $c^2x^4 - 2c^2x^3 + 2x - 1 = 0$, making $x = \sin\phi_1$, the equation for finding the required amplitude, which has only one real positive root.

166. As an instance of the application of the preceding results, we shall take the arc of the Lemniscata whose length

$$AP = cF_c(\phi), \text{ where } c = \frac{1}{\sqrt{2}},$$

and the $\angle\phi$ is such that $CP = \cos\phi$, supposing $AC = 1$, (Fig. 13.) see Example 4, Art. 138.

Let AQ, AB be two other arcs whose amplitudes are ψ and σ; then if

$$\cos\sigma = \cos\phi\cos\psi - \sin\phi\sin\psi\sqrt{1-\tfrac{1}{2}(\sin\sigma)^2},$$

we have
$$AB = AP + AQ;$$

from which fundamental equation, the following results may be deduced.

1. Let $\sigma = \frac{\pi}{2}$, then the radius vector $CB = 0$, and AB becomes the whole arc AQC; $\therefore AP = AC - AQ = CQ$; the amplitudes of AP, AQ, being connected by the equation

$$\tan\phi\tan\psi = \sqrt{2}, \text{ or } (\cos\phi)^2 = \frac{1-(\cos\psi)^2}{1+(\cos\psi)^2};$$

and consequently the radii vectóres $CP = \rho$, $CQ = \rho_1$, by the equation $\rho^2 = \frac{1-\rho_1{}^2}{1+\rho_1{}^2}$. If P and Q coincide, $\psi = \phi$,

$$\therefore (\tan\phi)^2 = \sqrt{2}, \text{ and } \cos\phi = \sqrt{2^{\frac{1}{2}}-1};$$

hence the whole arc is bisected in a point whose distance from C is $\sqrt{2^{\frac{1}{2}}-1}$.

2. Suppose P and Q to coincide, when CB is not $=0$; $\therefore\ AB = 2AP$; and the corresponding amplitudes are connected by the equation

$$\tan\frac{\sigma}{2} = \tan\phi\sqrt{1-\tfrac{1}{2}(\sin\phi)^2}, \text{ or } (\sin\phi)^2 = \frac{1-\cos\sigma}{1+\sqrt{1-\frac{1}{2}(\sin\sigma)^2}};$$

by means of which we can determine an arc that is double, or half, of a given arc. For instance, if ϕ be the amplitude of a quarter of the whole arc AC,

$$(\sin\phi)^2 = \frac{2^{\frac{1}{4}}-\sqrt{2-2^{\frac{1}{2}}}}{2^{\frac{1}{4}}+1},$$

$$\because\ \cos\sigma = \sqrt{2^{\frac{1}{2}}-1}, \text{ where } \sigma \text{ is the amplitude of half } AC.$$

3. Let the whole arc be trisected in P and Q, and ϕ, ψ, denote the amplitudes of AP, AQ; then, because AQ is bisected in P, $(\cos\phi)^2 - (\sin\phi)^2\sqrt{1-\frac{1}{2}(\sin\psi)^2} = \cos\psi$; and because $CQ = AP$, $\tan\phi\tan\psi = \sqrt{2}$; let $\cos\phi = x$,

$$\therefore\ (\cos\psi)^2 = \frac{1-x^2}{1+x^2},\ (\sin\psi)^2 = \frac{2x^2}{1+x^2},$$

hence, substituting and reducing,

$$\sqrt{1+x^2}-\sqrt{1-x^2} = \sqrt{1-x^4};\ \therefore\ x = \cos\phi = (2\sqrt{3}-3)^{\frac{1}{4}},$$

and the chord CP, subtending two-thirds of the whole arc, $= (2\sqrt{3}-3)^{\frac{1}{4}}$.

We proceed next to investigate formulæ for the comparison of elliptic functions of the second order; they all result from the following proposition.

167. The same relation among the amplitudes,

$$\cos\sigma = \cos\phi\cos\psi - \sin\phi\sin\psi\sqrt{1-c^2(\sin\sigma)^2},$$

$$\text{which gives } F(\phi)+F(\psi)-F(\sigma) = 0,$$

$$\text{gives } E(\phi)+E(\psi)-E(\cancel{\alpha}) = c^2\sin\phi\sin\psi\sin\sigma.$$

$E(\sigma)$

Since ψ is a function of ϕ by virtue of the equation of condition, we may assume

$$E(\phi) + E(\psi) - E(\sigma) = f(\phi);$$

$$\therefore\ d_\phi f(\phi) = \Delta(\phi) + \Delta(\psi) d_\phi \psi$$

$$= \frac{\cos\phi - \cos\psi\cos\sigma}{\sin\psi\sin\sigma} + \frac{\cos\psi - \cos\phi\cos\sigma}{\underset{\text{Sin}\,\phi}{\cancel{\sin\psi}}\sin\sigma} d_\phi\psi \text{ (Art. 158.)}$$

$$= \frac{1}{2\sin\phi\sin\psi\sin\sigma} d_\phi \{(\sin\phi)^2 + (\sin\psi)^2 + 2\cos\phi\cos\psi\cos\sigma\}.$$

But $(\sin\phi)^2 + (\sin\psi)^2 + 2\cos\phi\cos\psi\cos\sigma = 1 + (\cos\sigma)^2$

$$+ c^2 (\sin\phi\sin\psi\sin\sigma)^2,$$

$$\therefore\ d_\phi f(\phi) = c^2 d_\phi (\sin\phi\sin\psi\sin\sigma);$$

$$\therefore\ f(\phi) = c^2 \sin\phi\sin\psi\sin\sigma,$$

no constant being added $\because f(\phi)$ vanishes when $\phi = 0$. Hence,

$$E(\phi) + E(\psi) - E(\sigma) = c^2 \sin\phi\sin\psi\sin\sigma,$$

which is the general equation for the comparison of elliptic functions of the second order.

168. The different forms which the fundamental relation among the amplitudes,

$$\cos\sigma = \cos\phi\cos\psi - \sin\phi\sin\psi\sqrt{1 - c^2(\sin\sigma)^2}$$

was made to assume in Art. 159, in reference to the equation

$$F(\phi) + F(\psi) - F(\sigma) = 0,$$

are all of course applicable when ϕ, ψ, σ are considered as satisfying the equation

$$E(\phi) + E(\psi) - E(\sigma) = c^2 \sin\phi\sin\psi\sin\sigma,$$

or, more generally,

$$G(\phi) + G(\psi) - G(\sigma) = c^2 \sin\phi\sin\psi\sin\sigma,$$

denoting by $G(\phi)$, $E(\phi) + kF(\phi)$, k being any constant coefficient.

169. Make $\sigma = \frac{\pi}{2}$ in the equation

$$E(\phi) + E(\psi) - E(\sigma) = c^2 \sin\phi \sin\psi \sin\sigma,$$

$$\therefore\ E(\phi) + E(\psi) - E = c^2 \sin\phi \sin\psi,$$

and the equation connecting ϕ and ψ, becomes $b \tan\phi \tan\psi = 1$, or $\sin\phi = \frac{\cos\psi}{\Delta(\psi)}$;

$$\therefore\ E(\phi) + E(\psi) - E = \frac{c^2 \sin\psi \cos\psi}{\Delta(\psi)}, \text{ or } = \frac{c^2 \sin\phi \cos\phi}{\Delta(\phi)},$$

because ϕ and ψ are similarly involved.

If $\phi = \psi = \theta$, $2E(\theta) - E = c^2 (\sin\theta)^2 = c^2 \div \{1 + (\cot\theta)^2\}$

$$= \frac{c^2}{1+b} = 1 - b,\ \because (\tan\theta)^2 = \frac{1}{b};$$

$$\therefore E(\theta) = \tfrac{1}{2}E + \tfrac{1}{2}(1 - b),$$

and its complement $E - E(\theta) = \frac{1}{2}E - \frac{1}{2}(1 - b)$.

170. Again, in the general equation, let $\phi = \psi$,

$$\therefore\ 2E(\phi) - E(\sigma) = c^2 (\sin\phi)^2 \sin\sigma,$$

the relation between ϕ and σ being

$$\cos\sigma = (\cos\phi)^2 - (\sin\phi)^2 \sqrt{1 - c^2 (\sin\sigma)^2};$$

which, as in Art. 160, may be replaced by

$$\tan\frac{\sigma}{2} = \tan\phi\,\Delta(\phi), \text{ or } (\sin\phi)^2 = \frac{1 - \cos\sigma}{1 + \Delta(\sigma)};$$

the first gives the amplitude of a function that differs from twice a given function by an assignable algebraic quantity; and the second, one that differs from half a given function by an assignable algebraic quantity.

171. Lastly, if ϕ_2, ϕ_3, &c. be the amplitudes that satisfy the equations $2F(\phi_1) = F(\phi_2)$, $3F(\phi_1) = F(\phi_3)$, &c. the values of which may be obtained, Art. 164; and if in the equation

$$E(\phi_1) + E(\psi) - E(\sigma) = c^2 \sin\phi_1 \sin\psi \sin\sigma,$$

we change ψ into ϕ_1, ϕ_2, &c., ϕ_{n-1} successively; and, consequently, σ into ϕ_2, ϕ_3, &c., ϕ_n, by reason of the equation

$$F(\phi_1) + F(\psi) = F(\sigma);$$

by adding all the equations so formed together, we find

$$nE(\phi_1) - E(\phi_n) = c^2 \sin\phi_1 (\sin\phi_1 \sin\phi_2 + \sin\phi_2 \sin\phi_3 + \&\text{c.} + \sin\phi_{n-1} \sin\phi_n).$$

172. We shall now illustrate the above properties of elliptic functions of the second order, by certain problems on elliptic arcs.

Let the $\frac{1}{2}$ axes of the ellipse be 1 and b, and let $c = \sqrt{1 - b^2}$; (if it be desired that the $\frac{1}{2}$ axes should be a and b, it will only be necessary, by substituting $\frac{b}{a}$ instead of b, to make the results homogeneous); also let $x = \sin\phi$, and $y = b\cos\phi$, be the co-ordinates, parallel to the axes, of a point P in the ellipse, fig. 5; then, reckoning from the extremity of the axis minor, $BP = E_c(\phi)$, by Ex. 11, Art. 137.

Let BP, BK, BR be three arcs whose amplitudes ϕ, ψ, σ, or the abscissæ of whose extremities, $\sin\phi$, $\sin\psi$, $\sin\sigma$, are such, that

$$\cos\sigma = \cos\phi\cos\psi - \sin\phi\sin\psi\sqrt{1 - c^2(\sin\sigma)^2};$$

then $BP + BK - BR = c^2 \sin\phi \sin\psi \sin\sigma = c^2 . CN . CL . CM$,

which is the general equation for the comparison of elliptic arcs. The following are the principal results which may be deduced from it.

1. We can assign, in an infinite number of ways, two arcs whose difference is equal to an assignable straight line. For

$$BP - KR = BP + BK - BR = c^2 \sin\phi \sin\psi \sin\sigma;$$

$\therefore$ BP, KR have the required property; the angles ϕ and ψ may be assumed at pleasure, and then σ must be determined from the expression,

$$\sigma = \tan^{-1}\{\tan\phi\,\Delta(\psi)\} + \tan^{-1}\{\tan\psi\,\Delta(\phi)\}.$$

2. Let $\sigma = \frac{\pi}{2}$, then R coincides with A, and BR becomes the whole quadrant BA;

$$\therefore\; BP - KA = c^2 \sin\phi \sin\psi = c^2 . CN . CL,$$

under the condition $b\tan\phi\tan\psi = 1$;

$$\text{or}\quad BP - KA = \frac{c^2 \sin\phi\cos\phi}{\Delta(\phi)} = PY,$$

CY being perpendicular to the tangent at P;

$$\Big\{\text{for } CP^2 = (\sin\phi)^2 + b^2(\cos\phi)^2,\; CY^2 = b^2 \div (1 + b^2 - CP^2),$$

$$= b^2 \div \{1 - c^2(\sin\phi)^2\},\;\therefore\; PY = \frac{c^2\cos\phi\sin\phi}{\Delta(\phi)}\Big\};$$

this is Fagnani's theorem, and agrees with Ex. 12, Art. 137.

3. Let P and K coincide in the point O, R being at A, $\therefore$ $\psi = \phi = \theta$ suppose;

then $b(\tan\theta)^2 = 1$,

$$\therefore\; BO - AO = c^2(\sin\theta)^2 = \frac{c^2}{1+b} = 1 - b,$$

the co-ordinates of O being, $\sin\theta = \dfrac{1}{\sqrt{1+b}}$,

and $b\cos\theta = \dfrac{b^{\frac{3}{2}}}{\sqrt{1+b}}$, as found in Section VIII.

4. Let $\phi = \psi$ in the general equation, then K coincides with P,

$$\text{and } 2BP - BR = c^2 (\sin\phi)^2 \sin\sigma,$$

the equation of condition being

$$\cos\sigma = (\cos\phi)^2 - (\sin\phi)^2 \sqrt{1 - c^2 (\sin\sigma)^2}.$$

Hence if $\angle BCQ = \phi$, and we take

$$\tan \tfrac{1}{2} BCS = \tan \frac{\sigma}{2} = \tan\phi \Delta(\phi),$$

$$\text{then } BR = 2BP - c^2(\sin\phi)^2 \sin\sigma = 2BP - \frac{2c^2(\sin\phi)^3 \cos\phi \Delta(\phi)}{1 - c^2(\sin\phi)^4}.$$

But if BR be given, and the abscissa of P be taken

$$= \sin\phi = \sqrt{\frac{1 - \cos\sigma}{1 + \Delta(\sigma)}},$$

$$\text{then } BP = \tfrac{1}{2} BR + \tfrac{1}{2} \{1 - \Delta(\sigma)\} \tan\frac{\sigma}{2}.$$

Thus, let R coincide with the point O determined above,

$$\therefore\ (\sin\sigma)^2 = \frac{1}{1+b},\ \Delta(\sigma) = \sqrt{b},\ \tan\frac{\sigma}{2} = \sqrt{1+b} - \sqrt{b};$$

$$\begin{aligned} \therefore\ BP &= \tfrac{1}{2} BO + \tfrac{1}{2} (1 - \sqrt{b})(\sqrt{1+b} - \sqrt{b}) \\ &= \tfrac{1}{4} AB + \tfrac{1}{4} (1 - b) + \tfrac{1}{2} (1 - \sqrt{b})(\sqrt{1+b} - \sqrt{b}) \\ &= \tfrac{1}{4} AB + \tfrac{1}{4} (1 - \sqrt{b})(2\sqrt{1+b} + 1 - \sqrt{b}). \end{aligned}$$

173. We shall next exhibit the corresponding properties of hyperbolic arcs.

We have seen, (Ex. 14., Art. 137.) that if C be the center, and S the focus of the hyperbola AP, Fig. 4, and $CS = 1$, $CA = c$, $b = \sqrt{1 - c^2}$; and ϕ be such, that the ordinate

$$PN = b^2 \tan\phi, \text{ and consequently } CN = c \sec\phi \Delta(\phi),$$

$$CY = c\cos\phi,\ CP = \sqrt{c^2 + b^2(\tan\phi)^2},\ PY = \tan\phi \Delta(\phi);$$

then the length of the arc $AP = \tan\phi\,\Delta(\phi) - E(\phi) + b^2 F(\phi)$

$$= \tan\phi\,\Delta(\phi) - G(\phi),\ \{\text{if } G(\phi) \text{ denote } E(\phi) - b^2 F(\phi)\},$$

$$\text{or } PY - AP = G(\phi).$$

Let ϕ, ψ, σ be the amplitudes of three arcs, AP, AK, AR (and therefore $b^2\tan\phi$, $b^2\tan\psi$, $b^2\tan\sigma$ the ordinates of their extremities), connected by the equation

$$\cos\sigma = \cos\phi\cos\psi - \sin\phi\sin\psi\sqrt{1 - c^2(\sin\sigma)^2},$$

$$\text{then } G(\phi) + G(\psi) - G(\sigma) = c^2\sin\phi\sin\psi\sin\sigma,$$

which is the general equation for the comparison of hyperbolic arcs. The following are the principal results which may be deduced from it.

1. Let $\sigma = \frac{\pi}{2}$, and $\therefore\ b\tan\phi\tan\psi = 1$;

$$\therefore\ G(\phi) + G(\psi) - G = c^2\sin\phi\sin\psi,$$

$$\text{or } G + AP + AK = \tan\phi\,\Delta(\phi) + \tan\psi\,\Delta(\psi) - c^2\sin\phi\sin\psi$$

$$= \frac{\sin\phi}{\sin\psi} + \frac{\sin\psi}{\sin\phi} - c^2\sin\phi\sin\psi,$$

$$\left(\because\ \sin\phi = \frac{\cos\psi}{\Delta(\psi)},\ \sin\psi = \frac{\cos\phi}{\Delta(\phi)}\right)$$

$$\text{or } G + AP + AK = \frac{1}{\sin\phi\sin\psi},$$

$$\because\ (1 - c^2)(\sin\phi)^2(\sin\psi)^2 = \{1 - (\sin\phi)^2\}\{1 - (\sin\psi)^2\};$$

$$\therefore\ G + AP + AK = \frac{1}{c^2}CN\,.\,CL,\ (1),$$

$$\because\ \frac{1}{\sin\psi} = \Delta(\phi)\sec\phi = \frac{1}{c}CN,\ \frac{1}{\sin\phi} = \frac{1}{c}CL.$$

The complete function G represents the difference between the lengths of the asymptote and infinite hyperbolic arc (because when the amplitude $= \frac{1}{2}\pi$, the corresponding perpendicular upon the tangent vanishes), and is here expressed by means of two related arcs and the abscissæ of their extremities.

2. Let P and K coincide in the point O, R being at an infinite distance,

$$\therefore\ \phi = \psi = \theta \text{ suppose; then } b(\tan\theta)^2 = 1,$$

and the co-ordinates of O are

$$y = b^2 \tan\theta = b^{\frac{3}{2}},\quad x = c\sqrt{1+b};$$

also the intercept of the tangent at O

$$= \tan\theta\sqrt{1 - c^2(\sin\theta)^2} = 1,$$

$$\therefore\ G = 2G(\theta) - c^2(\sin\theta)^2 = 2(1 - AO) - (1 - b),$$

$$\text{or } G = 1 + b - 2AO.$$

The point O is a fixed point, and enjoys properties similar to to those of the point O in the ellipse.

3. Substitute this value of G in equation (1),

$$\therefore\ 1 + b - 2AO + AP + AK = \frac{1}{c^2} CN \,.\, CL,$$

$$\text{or } PO - KO = 1 + b - \frac{1}{c^2} CN \,.\, CL,$$

the points P, K being so related that the ordinate of O is a mean proportional between their ordinates; for the equation

$$b\tan\phi\tan\psi = 1 \text{ gives } b^2\tan\phi \,.\, b^2\tan\psi = b^3.$$

174. Elliptic functions of the second order can always be represented by the arcs of an ellipse; but to assign an algebraic curve whose arcs shall represent generally functions of the first order is more difficult. Perhaps the simplest curve which has this property is that which is constantly touched by a perpendicular to the diameter of an ellipse, through its extremity. Let ATB be this curve (fig. 11. bis) its tangent PT being perpendicular to CP; then if

$$\angle PTM = PCM = \theta, \text{ and } \therefore\ CP = \frac{b}{\sqrt{1 - c^2(\sin\theta)^2}},$$

$$\text{by Art. 136, arc } BT - TP = \int_\theta (CP) = bF(\theta),$$

because BT and θ vanish together:

$$\therefore\ BT = bF(\theta) + \frac{bc^2\sin\theta\cos\theta}{\Delta^3},\ \because\ PT = d_\theta(CP);$$

and making $\theta = \frac{1}{2}\pi$, the whole arc $ATB = bF$.

If $x = MT$, and $y = MC$ be co-ordinates of T, we have

$$x = CP\sin\theta + PT\cos\theta = \frac{b\sin\theta}{\Delta^3}(1 + c^2\cos 2\theta),$$

$$y = CP\cos\theta - PT\sin\theta = \frac{b\cos\theta}{\Delta^3}(b^2 + c^2\cos 2\theta),$$

by means of which equations the curve may be constructed.

175. As c varies from 0 to 1, $F_c(\phi)$ changes from $\int_\phi 1 = \phi$, to $\int_\phi \frac{1}{\cos\phi} = \log_e(\tan\phi + \sec\phi)$; similarly $E_c(\phi)$ changes from ϕ, to $\int_\phi \cos\phi = \sin\phi$, as c varies from 0 to 1. In Section VII. we expressed the values of these functions in series which converge with tolerable rapidity when c is small; we now proceed to investigate formulæ by which their values may be accurately computed for all values of c.

We shall first obtain the values of the complete functions F_c, E_c, when $c = 1$ very nearly, and therefore b is very small.

$$\text{Now } F_c(\phi) = \int_\phi \sec\phi\,\{1 + b^2(\tan\phi)^2\}^{-\frac{1}{2}}$$

$$= \int_\phi \sec\phi\,\left\{1 - \frac{1}{2}b^2(\tan\phi)^2 + \frac{1.3}{2.4}b^4(\tan\phi)^4 - \frac{1.3.5}{2.4.6}b^6(\tan\phi)^6 + \&c.\right\}$$

$$= \int_\phi \sec\phi - \frac{1}{2}b^2\left(\frac{1}{2}\sec\phi\tan\phi - \frac{1}{2}\int_\phi \sec\phi\right)$$

$$+ \frac{1.3}{2.4}b^4\left\{\frac{1}{4}\sec\phi\,(\tan\phi)^3 - \frac{3}{8}\sec\phi\tan\phi + \frac{3}{8}\int_\phi \sec\phi\right\} - \&c.;$$

$$\text{since } \int_\phi \sec\phi\,(\tan\phi)^n = \frac{1}{n}\sec\phi\,(\tan\phi)^{n-1}$$

$$-\frac{n-1}{n}\int_\phi \sec\phi\,(\tan\phi)^{n-2};$$

$$\therefore\ F_c(\phi) = \log_\epsilon(\tan\phi + \sec\phi).\left(1 + \frac{b^2}{4} + \frac{1.3}{2.4}.\frac{3}{8}b^4 + \&c.\right)$$

$$-\frac{b^2}{4}\sec\phi\tan\phi\left\{1 - \frac{1.3}{2.4}b^2(\tan\phi)^2 + \frac{1.3}{2.4}.\frac{3}{2}b^2 + \&c.\right\}.$$

Let $\tan\phi = \frac{1}{\sqrt{b}}$, $\therefore\ F_c(\phi) = \frac{1}{2}F_c$, and $\sec\phi = \frac{\sqrt{1+b}}{\sqrt{b}}$;

$$\therefore\ F_c = 2\log_\epsilon\left(\frac{1+\sqrt{1+b}}{\sqrt{b}}\right)\left(1 + \frac{b^2}{4} + \frac{1.3}{2.4}.\frac{3}{8}b^4 + \&c.\right)$$

$$-\frac{b\sqrt{1+b}}{2}\left(1 - \frac{1.3}{2.4}b + \frac{1.3}{2.4}\,\frac{3}{2}b^2 + \&c.\right).$$

Hence when $b = 0$ very nearly, or $c = 1$,

$$F_c = 2\log_\epsilon\left(\frac{2}{\sqrt{b}}\right) = \log_\epsilon\left(\frac{4}{b}\right).$$

176. Similarly, $E_c(\phi) = \int_\phi \cos\phi\,\{1 + b^2(\tan\phi)^2\}^{\frac{1}{2}}$

$$= \int_\phi \cos\phi\left\{1 + \frac{1}{2}b^2(\tan\phi)^2 - \frac{1.1}{2.4}b^4(\tan\phi)^4 + \&c.\right\}$$

$$= \sin\phi\left\{1 - \frac{1}{2}b^2 - \frac{1.1}{2.4}.\frac{1}{2}b^4(\tan\phi)^2 - \frac{1.1}{2.4}.\frac{3}{2}b^4 + \&c.\right\}$$

$$+ \log_\epsilon(\tan\phi + \sec\phi)\left(\frac{b^2}{2} + \frac{1.1}{2.4}.\frac{3}{2}b^4 + \&c.\right),$$

by the formula $\int_\phi \cos\phi\,(\tan\phi)^n$

$$= \frac{1}{n-2}\sin\phi\,(\tan\phi)^{n-3} - \frac{n-1}{n-2}\int_\phi \cos\phi\,(\tan\phi)^{n-2};$$

$$\text{let } \tan\phi = \frac{1}{\sqrt{b}} \quad \therefore\ E_c(\phi) = \frac{1}{2}E_c + \frac{1}{2}(1-b);$$

$$\therefore\ E_c = -1 + b + \frac{2}{\sqrt{1+b}}\left\{1 - \frac{1}{2}b^2 - \frac{1.1}{2.4}\cdot\frac{b^3}{2} + \&\text{c.}\right\}$$

$$+ 2\log_e\left(\frac{1+\sqrt{1+b}}{\sqrt{b}}\right)\left(\frac{b^2}{2} + \frac{1.1}{2.4}\cdot\frac{3}{2}b^4 + \&\text{c.}\right).$$

Hence when $c = 1$ very nearly, or $b = 0$,

$$E_c = 1 + \frac{b^2}{2}\left\{\log_e\left(\frac{4}{b}\right) - \frac{1}{2}\right\}.$$

177. In order to approximate to the value of $F_c(\phi)$ for any values of c and ϕ, we must change it into another similar function having a different modulus and amplitude. By the transformation of Lagrange which we are about to give, we may compute either a scale of decreasing moduli reducing $F_c(\phi)$ to an angle, or a scale of increasing moduli bringing it nearer to a logarithm; the process depends upon the following proposition.

178. If the amplitudes and moduli of two elliptic functions of the first order $F_c(\phi)$ and $F_{c_1}(\phi_1)$, be connected respectively by the equations

$$\sin(2\phi - \phi_1) = c_1 \sin\phi_1, \quad c = \frac{2\sqrt{c_1}}{1+c_1},$$

the functions themselves are connected by the equation

$$F_c(\phi) = \frac{1+c_1}{2}F_{c_1}(\phi_1).$$

The equation $\sin(2\phi - \phi_1) = c_1\sin\phi_1$,

$$\text{gives } \cos(2\phi - \phi_1) = \sqrt{1 - (c_1\sin\phi_1)^2} = \Delta_1;$$

$$\therefore\ \cos 2\phi = \cos(2\phi - \phi_1 + \phi_1) = \Delta_1\cos\phi_1 - c_1(\sin\phi_1)^2,$$

$$\text{or } (\sin\phi)^2 = \tfrac{1}{2}\left\{1 - \Delta_1\cos\phi_1 + c_1(\sin\phi_1)^2\right\},$$

$$\therefore\ 1-(c\sin\phi)^2 = 1-\frac{2c_1}{(1+c_1)^2}\left\{1-\Delta_1\cos\phi_1+c_1(\sin\phi_1)^2\right\}$$

$$=\frac{1}{(1+c_1)^2}\left\{1+c_1^2+2c_1\Delta_1\cos\phi_1-2(c_1\sin\phi_1)^2\right\}$$

$$=\frac{\Delta_1^2+2\Delta_1c_1\cos\phi_1+(c_1\cos\phi_1)^2}{(1+c_1)^2};$$

$$\therefore\ \Delta=\frac{\Delta_1+c_1\cos\phi_1}{1+c_1};\quad \text{also } 2\phi=\phi_1+\sin^{-1}(c_1\sin\phi_1),$$

$$\therefore\ 2d_{\phi_1}(\phi)=1+\frac{c_1\cos\phi_1}{\Delta_1}=\frac{\Delta_1+c_1\cos\phi_1}{\Delta_1};$$

$$\therefore\ \frac{2}{\Delta}d_{\phi_1}(\phi)=\frac{1+c_1}{\Delta_1},\quad \therefore\ \int_\phi\frac{1}{\Delta}=\frac{1+c_1}{2}\int_{\phi_1}\frac{1}{\Delta_1},$$

$$\text{or } F_c(\phi)=\frac{1+c_1}{2}F_{c_1}(\phi_1),$$

since ϕ and ϕ_1 vanish together.

179. The above relations may be expressed differently, as follows.

The equation $c=\dfrac{2\sqrt{c_1}}{1+c_1}$, gives $b=\sqrt{1-c^2}=\dfrac{1-c_1}{1+c_1}$,

and therefore $c_1=\dfrac{1-b}{1+b}$; also $\dfrac{\sin\phi_1}{\sin(2\phi-\phi_1)}=\dfrac{1}{c_1}$, gives

$$\frac{\tan(\phi_1-\phi)}{\tan\phi}=\frac{1-c_1}{1+c_1},\ \text{or } \tan(\phi_1-\phi)=b\tan\phi.$$

$$\text{Hence } F_c(\phi)=\frac{1}{1+b}F_{c_1}(\phi_1).$$

If $\phi=\dfrac{\pi}{2}$, then $\phi_1=\pi$, and (Art. 99.) $F_{c_1}(\pi)=2F_{c_1}$;

hence the relation between the complete functions is

$$F_c = (1 + c_1) F_{c_1}, \text{ or } F_c = \frac{2}{1+b} F_{c_1}.$$

180. Hence to transform a function $F_c(\phi)$ into another with a smaller modulus, we have

$$F_c(\phi) = \frac{1}{1+b} F_{c_1}(\phi_1);$$

c_1, ϕ_1, being determined by the equations

$$c_1 = \frac{1-b}{1+b}, \text{ tan } (\phi_1 - \phi) = b \tan \phi;$$

(c_1 is manifestly less than c, $\because \frac{c_1}{c} = \frac{c}{(1+\sqrt{1-c^2})^2}$ which is < 1; and $\phi_1 > \phi$);

and for a complete function, $F_c = \frac{2}{1+b} F_{c_1}$.

To transform a function $F_{c_1}(\phi_1)$ into another with a larger modulus, we have

$$F_{c_1}(\phi_1) = \frac{2}{1+c_1} F_c(\phi);$$

c, ϕ, being determined by the equations

$$c = \frac{2\sqrt{c_1}}{1+c_1}, \text{ sin } (2\phi - \phi_1) = c_1 \sin \phi_1;$$

and for a complete function, $F_{c_1} = \frac{1}{1+c_1} F_c$.

The first transformation is to be used when $c < \sin 45^0$, so that by repeated applications the function may be reduced to an angle; and the second when $c > \sin 45^0$, in which case the function will be brought up to a logarithm: the two following Articles contain the resulting approximations.

181. To approximate to the value of $F_c(\phi)$ to any degree of accuracy, c being $< \sin 45^0$.

Here we must reduce $F_c(\phi)$ to an angle, by diminishing the modulus.

Let there be a series of decreasing moduli c, c_1, c_2, &c., c_n derived from one another by the law that

$$c_1 = \frac{1-b}{1+b},\ c_2 = \frac{1-b_1}{1+b_1},\ \&c,\ c_n = \frac{1-b_{n-1}}{1+b_{n-1}};$$

b, b_1, &c., b_{n-1} denoting the complements of c, c_1, &c., c_{n-1};

and a series of increasing amplitudes ϕ, ϕ_1, ϕ_2, &c., ϕ_n, such that

$$\tan(\phi_1 - \phi) = b\tan\phi, \quad \tan(\phi_2 - \phi_1) = b_1 \tan\phi_1, \ \&c.$$

$$\tan(\phi_n - \phi_{n-1}) = b_{n-1}\tan\phi_{n-1};$$

$$\therefore\ F_c(\phi) = \frac{1}{1+b}F_{c_1}(\phi_1) = \frac{1}{1+b}\cdot\frac{1}{1+b_1}\cdot F_{c_2}(\phi_2)$$

$$= \frac{1}{1+b}\cdot\frac{1}{1+b_1}\cdot\frac{1}{1+b_2}\cdots\frac{1}{1+b_{n-1}}F_{c_n}(\phi_n).$$

As the moduli decrease rapidly, a small number of operations will give a sufficiently accurate result. If n be so large that we may suppose $c_n = 0$, then $F_{c_n}(\phi_n) = \phi_n$.

When the process is pushed so far that the modulus is very small, the modulus and amplitude of the succeeding function may be obtained by the series

$$c_{r+1} = \frac{1-\sqrt{1-c_r^2}}{1+\sqrt{1-c_r^2}} = \tfrac{1}{4}c_r^2 + \frac{1.3}{4.6}c_r^4 + \frac{1.3.5}{4.6.8}c_r^6 + \&c.,$$

$$\phi_{r+1} = 2\phi_r - c_r\sin 2\phi_r + \tfrac{1}{2}c_r^2\sin 4\phi_r - \tfrac{1}{3}c_r^3\sin 6\phi_r + \&c.$$

For the value of the complete function we have

$$F_c = (1+c_1)\,F_{c_1} = (1+c_1)(1+c_2)F_{c_2} = (1+c_1)(1+c_2)\dots(1+c_n)\,\frac{\pi}{2},$$

the number of factors being such that $c_n = 0$, very nearly.

182. To approximate to the value of $F_c(\phi)$, to any degree of accuracy, c being $>\sin 45^0$.

Here we must raise $F_c(\phi)$ to a logarithm, by increasing the modulus.

Let there be a series of increasing moduli c, c', c'', &c. $c^{(n)}$ derived from one another by the law that

$$c' = \frac{2\sqrt{c}}{1+c}, \quad c'' = \frac{2\sqrt{c'}}{1+c'}, \text{ \&c.,} \quad c^{(n)} = \frac{2\sqrt{c^{(n-1)}}}{1+c^{(n-1)}};$$

and a series of decreasing amplitudes $\phi, \phi', \phi'', \phi'''$, &c. $\phi^{(n)}$, such that

$$\sin(2\phi' - \phi) = c\sin\phi, \quad \sin(2\phi'' - \phi') = c'\sin\phi', \text{ \&c.,}$$
$$\sin\left(2\phi^{(n)} - \phi^{(n-1)}\right) = c^{(n-1)}\sin\phi^{(n-1)}.$$

Then $F_c(\phi) = \dfrac{2}{1+c}F_{c'}(\phi') = \dfrac{2}{1+c}\cdot\dfrac{2}{1+c'}F_{c''}(\phi'')$

$$= \frac{2}{1+c}\cdot\frac{2}{1+c'}\cdot\frac{2}{1+c''}\dots\frac{2}{1+c^{(n-1)}}F_{c^{(n)}}(\phi^{(n)}).$$

If n be so great that we may suppose $c^{(n)} = 1$,

then $F_{c^{(n)}}(\phi^{(n)}) = \log_\epsilon \tan\left(\dfrac{\pi}{4} + \dfrac{\phi^{(n)}}{2}\right)$.

For the value of the complete function we have

$$F_c = \frac{1}{1+c}F_{c'} = \frac{1}{1+c}\cdot\frac{1}{1+c'}F_{c''}$$

$$= \frac{1}{(1+c)(1+c')\dots(1+c^{(n-1)})}\log_\epsilon\left(\frac{4}{b^{(n)}}\right) \text{ (Art. 175).}$$

We shall next apply Lagrange's transformation to functions of the second order.

183. Let there be two elliptic functions of the second order $E_c(\phi)$ and $E_{c_1}(\phi_1)$, whose moduli and amplitudes are connected by the same equations as in Art. 178, viz.

$$c = \frac{2\sqrt{c_1}}{1 + c_1}, \text{ and } \sin(2\phi - \phi_1) = c_1 \sin\phi_1 ;$$

then, since $(1 + c_1)\Delta = \Delta_1 + c_1 \cos\phi_1$, and

$$2 d_{\phi_1}(\phi) = \frac{\Delta_1 + c_1 \cos\phi_1}{\Delta_1}, \quad \therefore\ 2(1 + c_1)\Delta d_{\phi_1}(\phi) = \frac{(\Delta_1 + c_1 \cos\phi_1)^2}{\Delta_1}$$

$$= \frac{1}{\Delta_1}\left\{\Delta_1^2 + 2\Delta_1 c_1 \cos\phi_1 + c_1^2 - c_1^2 (\sin\phi_1)^2\right\}$$

$$= \frac{1}{\Delta_1}(2\Delta_1^2 + 2\Delta_1 c_1 \cos\phi_1 - b_1^2) = 2\Delta_1 + 2c_1 \cos\phi_1 - \frac{b_1^2}{\Delta_1};$$

therefore, integrating,

$$(1 + c_1) E_c(\phi) = E_{c_1}(\phi_1) + c_1 \sin\phi_1 - \tfrac{1}{2} b_1^2 F_{c_1}(\phi_1),$$

the equation which connects the two functions; where, as before, the equations connecting the moduli and amplitudes may be replaced by

$$c_1 = \frac{1 - b}{1 + b}, \text{ and } \tan(\phi_1 - \phi) = b \tan\phi.$$

By this formula, a function of the first order is expressed by means of two functions of the second order.

For complete functions, making $\phi = \frac{\pi}{2}$, $\phi_1 = \pi$, we have

$$(1 + c_1) E_c = 2E_{c_1} - b_1^2 F_{c_1}, \quad \text{or } E_c = (1 + b) E_{c_1} - (1 - c_1) F_{c_1}.$$

184. By the above formula of reduction, $E_c(\phi)$ is expressed by functions of the first and second orders having smaller moduli; and the operation may be continued at pleasure.

Thus, let $E_{c_2}(\phi_2)$ be another function whose modulus and amplitude are such that

$$c_2 = \frac{1 - b_1}{1 + b_1}, \text{ and } \tan(\phi_2 - \phi_1) = b_1 \tan\phi_1,$$

$$\therefore\ (1 + c_2) E_{c_1}(\phi_1) = E_{c_2}(\phi_2) + c_2 \sin\phi_2 - \tfrac{1}{2}(b_2)^2 F_{c_2}(\phi_2);$$

therefore, by division, {denoting $E_c(\phi)$, $E_{c_1}(\phi_1)$, $E_{c_2}(\phi_2)$, by E, E_1, E_2, for the sake of brevity},

$$\frac{(1 + c_1) E - E_1 - c_1 \sin\phi_1}{(1 + c_2) E_1 - E_2 - c_2 \sin\phi_2} = \left(\frac{b_1}{b_2}\right)^2 \cdot \frac{F_1}{F_2}$$

$$= \frac{{b_1}^2}{1 - {c_2}^2} \cdot \frac{1 + c_2}{2} = \frac{{b_1}^2}{2} \cdot \frac{1}{1 - c_2}, \text{ by Art. 178;}$$

$$\text{but } c_2 = \frac{1 - b_1}{1 + b_1}, \quad \therefore\ 1 + c_2 = \frac{2}{1 + b_1}, \quad 1 - c_2 = \frac{2b_1}{1 + b_1},$$

$$\therefore\ (1 + c_1) E - E_1 - c_1 \sin\phi_1$$

$$= \frac{b_1}{4}(1 + b_1)\{(1 + c_2) E_1 - E_2 - c_2 \sin\phi_2\}$$

$$= \frac{b_1}{4}\{2E_1 - (1 + b_1) E_2 - (1 - b_1) \sin\phi_2\};$$

$$\therefore\ 2(1 + c_1) E - (2 + b_1) E_1 + \frac{b_1}{2}(1 + b_1) E_2 - 2c_1 \sin\phi_1$$

$$+ \frac{b_1(1 - b_1)}{2} \sin\phi_2 = 0,$$

the equation connecting three consecutive functions of the second order.

Let $\phi = \frac{\pi}{2}$, and therefore, $\phi_1 = \pi$, $\phi_2 = 2\pi$; the formula then becomes,

$$2(1+c_1)E_c - (2+b_1)E_{c_1}(\pi) + \frac{b_1}{2}(1+b_1)E_{c_2}(2\pi) = 0,$$

$$\text{or } (1+c_1)E_c - (2+b_1)E_{c_1} + b_1(1+b_1)E_{c_2} = 0,$$

$$\text{since } E_{c_1}(\pi) = 2E_{c_1}, \quad E_{c_1}(2\pi) = 4E_{c_1}.$$

185. The formula of Art. 183. as was observed, enables us to express an elliptic function of the first order, by two elliptic functions of the second order; if the modulus and amplitude of a proposed function of the first order be c_1, ϕ_1, we have

$$b_1^2 F_{c_1}(\phi_1) = 2E_{c_1}(\phi_1) - 2(1+c_1)E_c(\phi) + 2c_1 \sin\phi_1,$$

c and ϕ being given by the equations

$$c = \frac{2\sqrt{c_1}}{1+c_1}, \quad \sin(2\phi - \phi_1) = c_1 \sin\phi_1.$$

Hence a hyperbolic arc can be expressed by two elliptic arcs, which is Landen's theorem.

For by Ex. 14, Art. 137, if in Fig. 4. we make

$$CS = 1, \quad CA = c_1, \quad \text{and } NP = b_1^2 \tan\phi_1,$$

length of hyperbolic arc AP

$$= \tan\phi_1 \sqrt{1 - c_1^2(\sin\phi_1)^2} - E_{c_1}(\phi_1) + b_1^2 F_{c_1}(\phi_1)$$

$$= \tan\phi_1 \sqrt{1 - c_1^2(\sin\phi_1)^2} + E_{c_1}(\phi_1) - 2(1+c_1)E_c(\phi) + 2c_1 \sin\phi_1.$$

186. Hence also we can express the difference between the lengths of the asymptote and infinite hyperbolic arc, by the difference between the lengths of two elliptic quadrants.

$$\text{For } PY - AP = 2(1+c_1)E_c(\phi) - E_{c_1}(\phi_1) - 2c_1 \sin\phi_1.$$

$$\text{Let } \phi_1 = \frac{\pi}{2}, \quad \therefore \cos 2\phi = -c_1, \text{ and consequently,}$$

$$(\sin\phi)^2 = \frac{1+c_1}{2} = \frac{1}{1+b}; \quad \therefore \text{(Art. 169.)}\ E_c(\phi) = \tfrac{1}{2}E_c + \frac{1-b}{2};$$

hence the difference between the asymptote and infinite hyperbolic arc

$$= (1+c_1)E_c + (1+c_1)(1-b) - E_{c_1} - 2c_1$$

$$= (1+c_1)E_c - E_{c_1}, \quad \left(\because c_1 = \frac{1-b}{1+b}\right)$$

$= \text{length of elliptic quadrant } \{\tfrac{1}{2} \text{ axes, } (1+c_1) \text{ and } (1-c_1)\}$

$- \text{length of elliptic quadrant } \{\tfrac{1}{2} \text{ axes, } 1 \text{ and } \sqrt{1-c_1^2}\}.$

187. The formula of Art. 184. enables us to assign the value of $E_c(\phi)$ in terms of two similar functions having smaller moduli, and by continuing the process, we may obtain a result as accurate as we please; we may, however, approximate to this value more conveniently, by considering the more general form

$$G_c(\phi) = \int_\phi \frac{a_0 + b_0(\sin\phi)^2}{\Delta_c(\phi)};$$

then, making the same assumptions as in Art. 178,

$$(\sin\phi)^2 = \tfrac{1}{2}\{1 - \Delta_1\cos\phi_1 + c_1(\sin\phi_1)^2\}, \text{ and } \frac{1}{\Delta}d_{\phi_1}(\phi) = \frac{1+c_1}{2}\cdot\frac{1}{\Delta_1},$$

$$\therefore\ G_c(\phi) = \int_{\phi_1}\{a_0 + \frac{b_0}{2}(1 - \Delta_1\cos\phi_1 + c_1(\sin\phi_1)^2)\}\frac{1+c_1}{2}\cdot\frac{1}{\Delta_1}$$

$$= \frac{1+c_1}{2}\left\{\int_{\phi_1}\left(a_0 + \frac{b_0}{2} + \frac{b_0c_1}{2}(\sin\phi_1)^2\right)\frac{1}{\Delta_1} - \frac{b_0}{2}\sin\phi_1\right\}$$

$$= \frac{1+c_1}{2}\left\{\int_{\phi_1}(a_1 + b_1(\sin\phi_1)^2)\frac{1}{\Delta_1} - \frac{b_0}{2}\sin\phi_1\right\},$$

$$\text{if } a_1 = a_0 + \frac{b_0}{2}, \quad b_1 = \frac{b_0c_1}{2};$$

$$\text{or } G_c(\phi) = \frac{1+c_1}{2}\left\{G_{c_1}(\phi_1) - \tfrac{1}{2}b_0 \sin\phi_1\right\};$$

$$\text{similarly, } G_{c_1}(\phi_1) = \frac{1+c_2}{2}\left\{G_{c_2}(\phi_2) - \tfrac{1}{2}b_1 \sin\phi_2\right\}, \text{ \&c.}=\text{\&c.};$$

$$G_{c_{n-1}}(\phi_{n-1}) = \frac{1+c_n}{2}\left\{G_{c_n}(\phi_n) - \tfrac{1}{2}b_{n-1} \sin\phi_n\right\}.$$

Let n be of such a magnitude that c_n may be neglected;

$\therefore\ \Delta_n=1$; also $b_1=\frac{1}{2}b_0c_1$, $b_2=\frac{1}{2}b_1c_2=\frac{1}{4}c_1c_2b_0$, $b_3=\frac{1}{2}b_2c_3=\frac{1}{8}c_1c_2c_3b_0$,

and $b_n=\frac{1}{2^n}c_1c_2\ldots c_nb_0$; therefore, *a fortiori*, b_n may be neglected;

and $a_1=a_0+\frac{1}{2}b_0$, $a_2=a_1+\frac{1}{2}b_1=a_0+\frac{1}{2}b_0+\frac{1}{2}b_1$,

$a_3=a_2+\frac{1}{2}b_2=a_0+\frac{1}{2}b_0+\frac{1}{2}b_1+\frac{1}{2}b_2$, $a_n=a_0+\frac{1}{2}b_0+\frac{1}{2}b_1+\frac{1}{2}b_2+\text{\&c.}+\frac{1}{2}b_{n-1}$

$$=a_0+\tfrac{1}{2}b_0\left(1+\tfrac{1}{2}c_1+\tfrac{1}{4}c_1c_2+\tfrac{1}{8}c_1c_2c_3+\text{\&c.}+\frac{1}{2^{n-1}}c_1c_2\ldots c_{n-1}\right);$$

$$\text{and } G_{c_n}(\phi_n)=\int_{\phi_n}\left\{a_n+b_n(\sin\phi_n)^2\right\}=a_n\phi_n.$$

Hence, collecting the results, and substituting for a_n, b_1, b_2, &c., b_{n-1}, their values, we have

$$G_c(\phi)=\frac{(1+c_1)(1+c_2)\ldots(1+c_n)}{2^n}\left\{a_0+\tfrac{1}{2}b_0\left(1+\tfrac{1}{2}c_1+\tfrac{1}{4}c_1c_2+\text{\&c.}\right.\right.$$

$$\left.\left.+\frac{1}{2^{n-1}}c_1c_2\ldots c_{n-1}\right)\right\}\phi_n-\tfrac{1}{2}b_0\left\{\frac{1+c_1}{2}\sin\phi_1+\frac{(1+c_1)(1+c_2)}{4}\right.$$

$$\cdot\tfrac{1}{2}c_1\sin\phi_2+\frac{(1+c_1)(1+c_2)(1+c_3)}{8}\cdot\tfrac{1}{4}c_1c_2\sin\phi_3+\text{\&c.}$$

$$\left.+\frac{(1+c_1)(1+c_2)\ldots(1+c_n)}{2^n}\cdot\frac{1}{2^{n-1}}c_1c_2\ldots c_{n-1}\sin\phi_n\right\};$$

the moduli and amplitudes being derived from one another by the formulæ

$$c_1=\frac{1-b}{1+b},\quad \tan(\phi_1-\phi)=b\tan\phi.$$

For complete functions, $\phi = \frac{\pi}{2}$, $\phi_1 = \pi$,

$$\therefore\ G_c = (1+c_1)\,G_{c_1} = (1+c_1)(1+c_2)\,G_{c_2} = (1+c_1)(1+c_2)\ldots(1+c_n)\,\frac{\pi}{2}\,a_n,$$

$$\text{or } G_c = \frac{\pi}{2}(1+c_1)(1+c_2)\ldots(1+c_n)\left\{a_0 + \tfrac{1}{2}b_0\left(1 + \tfrac{1}{2}c_1 + \tfrac{1}{4}c_1c_2 + \&c.\right.\right.$$

$$\left.\left. + \frac{1}{2^{n-1}}c_1c_2\ldots c_{n-1}\right)\right\}.$$

188. If we make $a_0 = 1$, $b_0 = -c^2$, $G_c(\phi)$ will become $= E_c(\phi)$, and the resulting formulæ will give the length of any elliptic arc. Also by Ex. 14, Art. 137,

length of a hyperbolic arc

$$= \tan\phi\sqrt{1-c^2(\sin\phi)^2} - \int_\phi \frac{c^2(\sin\phi)^2}{\sqrt{1-c^2(\sin\phi)^2}},$$

hence, if in the value of $G_c(\phi)$ we make $a_0 = c^2$, $b_0 = -c^2$, we obtain the length of any hyperbolic arc.

189. The transformation of Lagrange, is a particular case of the general one of Jacobi, which we shall now give; but we must first explain the following notation, which will be found convenient in investigating it.

Let $F_k(\phi)$ be the function which it is proposed to transform, of known modulus k, and amplitude ϕ which may have any value from zero to infinity; and let it be denoted by u, so that

$$F_k(\phi) = u;$$

$$\therefore\ \phi = \text{amplitude of } u = Au, \text{ as it is written,}$$

$$\text{and } \sin\phi = \sin Au.$$

Let $\frac{1}{p}F_k = \omega$, p being any given whole number;

and let a_1, a_2, a_3, &c. be the amplitudes of ω, 2ω, 3ω, &c.

so that

$$F_k(a_1) = \frac{1}{p} F_k,\ F_k(a_2) = \frac{2}{p} F_k = 2F_k(a_1),\ F_k(a_3) = 3F_k(a_1),\ \&c.;$$

then a_1 may be approximated to by Art. 165, and a_2, a_3, &c. obtained from a_1, by Art. 164; therefore a_1, a_2, a_3, &c. may be regarded as known in terms of p and k.

$$\text{Then } p\omega = F_k,\ \therefore\ A(p\omega) = \frac{\pi}{2},\ \text{and } \sin A(p\omega) = \sin a_p = 1;$$

$$2p\omega = 2F_k = F_k(\pi),\ \therefore\ A(2p\omega) = \pi,\ \sin A(2p\omega) = 0;$$

$$np\omega = nF_k = F_k\left(\frac{n\pi}{2}\right),\ \therefore\ A(np\omega) = \frac{n\pi}{2},$$

$$\sin A(np\omega) = 0, \text{ or } \pm 1,$$

according as n is even or odd.

190. Again, recalling the property of all such integrals as $F_k(\phi)$, viz.

$$F_k(n\pi \pm \phi) = F_k(n\pi) \pm F_k(\phi),$$

n being a whole number, (Art. 99),

we have, if $2r+1$ be any odd number less than $2p$,

$$(2p-2r-1)\omega = 2F_k - \frac{2r+1}{p} F_k = F_k(\pi) - F_k(a_{2r+1}) = F_k(\pi - a_{2r+1}),$$

$$\therefore\ \sin a_{2p-2r-1} = \sin A(2p-2r-1)\omega = \sin a_{2r+1}.$$

$$\text{Also, } 2p\omega \pm u = F_k(\pi) \pm F_k(\phi) = F_k(\pi \pm \phi),$$

$$\therefore\ \sin A(2p\omega \pm u) = \mp \sin\phi = \mp \sin Au;$$

$$u + 2np\omega = F_k(\phi) + F_k(n\pi) = F_k(n\pi + \phi),$$

$$\therefore\ \sin A(u + 2np\omega) = (-1)^n \sin Au.$$

191. Hence, the formula of Art. 159,

$$\sin\sigma\sin\delta = \frac{(\sin\phi)^2 - (\sin\psi)^2}{1 - (k\sin\phi\sin\psi)^2},$$

may for the present purpose, be more conveniently expressed.

For let $F_k(\phi) = u,\ F_k(\psi) = v,$

$$\therefore\ \sigma = A(u+v),\ \delta = A(u-v),$$

$$\therefore\ \sin A(u+v)\sin A(u-v) = \frac{(\sin Au)^2 - (\sin Av)^2}{1 - (k\sin Au\sin Av)^2}.$$

We observe that this product has its greatest value, $(\sin\phi)^2$, when $v = 0$; for if any other value be subtracted from $(\sin\phi)^2$, the result is positive.

Suppose $v = n\omega$, so that $Av = a_n$,

$$\therefore\ \sin A(u+n\omega)\sin A(u-n\omega) = \frac{(\sin\phi)^2 - (\sin a_n)^2}{1 - (k\sin\phi\sin a_n)^2}.$$

192. We now come to the enunciation of Jacobi's first Theorem.

If the amplitudes of two elliptic functions of the first order $F_k(\phi)$, $F_h(\psi)$, are connected by the equation

$$y = \beta x\frac{1 - (x\operatorname{cosec} a_2)^2}{1 - (kx\sin a_2)^2}\,\frac{1 - (x\operatorname{cosec} a_4)^2}{1 - (kx\sin a_4)^2}\cdots\frac{1 - (x\operatorname{cosec} a_{p-1})^2}{1 - (kx\sin a_{p-1})^2},$$

where $x = \sin\phi$, $y = \sin\psi$, p is any odd integer, and

$$\beta = \left(\frac{\sin a_2\sin a_4\ldots\sin a_{p-1}}{\sin a_1\sin a_3\ldots\sin a_{p-2}}\right)^2,$$

a_1, a_2, a_3, &c. being the amplitudes of

$$\frac{1}{p}F_k,\ \frac{2}{p}F_k,\ \frac{3}{p}F_k,\ \text{\&c.};$$

and the moduli of the functions by the equation

$$h = k^p(\sin a_1\sin a_3\ldots\sin a_{p-2})^4;$$

the functions themselves are connected by the equation

$$F_h(\psi) = \beta F_k(\phi).$$

193. The object of this theorem is to transform $F_k(\phi)$ into $\frac{1}{\beta}F_h(\psi)$, β and h being unknown constants depending upon k, and ψ upon k and ϕ; and its demonstration is founded upon the analogy which an elliptic function bears to an angle; since, when the modulus vanishes, the function becomes equal to its amplitude. Now we shall make a supposition with respect to each of the unknown constants, viz., that h becomes 0, and β becomes a whole number p, when $k = 0$, (it will be readily seen that these suppositions agree with the values of h and β in the above enunciation, observing that when $k = 0$, $F_k = \frac{\pi}{2}\Big)$; therefore when $k = 0$, we shall have $\psi = p\phi$; if therefore any algebraic relation can be obtained between ψ and ϕ, it must be such as, when $k = 0$, to coincide with the relation between two angles, one of which is a multiple of the other, *i. e.* taking the usual formula, and putting $\frac{\pi}{2p} = a$, with

$$\sin\psi = \frac{\sin\phi \sin(\phi + 2a)\sin(\phi + 4a)\ldots\sin\{\phi + 2(p-1)a\}}{\sin a \sin 3a \sin 5a \ldots \sin(2p-1)a}.$$

Hence, we are led to consider the properties of the function of the amplitude denoted by y in the following Art., which, (since $Au = \phi$, and $A\omega = \omega = \frac{1}{p}\frac{\pi}{2} = a$, when $k = 0$) it will be observed is formed upon the model of that which effects the transformation in the particular case when the functions become angles, and coincides with it when the moduli vanish.

194. Let

$$y = \frac{\sin A(u) \sin A(u+2\omega) \sin A(u+4\omega)\ldots\sin A\{u+2(p-1)\omega\}}{\sin A(\omega)\sin A(3\omega)\sin A(5\omega)\ldots\sin A(2p\omega-\omega)};$$

we must first shew that y represents the sine of an angle, and determine the law of its increase consequent upon the increase of $A(u)$ or ϕ.

If in the numerator we change u into $u+2\omega$, it becomes

$$-\sin A(u+2\omega)\sin A(u+4\omega)\ldots\sin A\left\{u+2(p-1)\omega\right\}\sin A(u),$$

$\because$ the last factor is $\sin A(u+2p\omega)$, which equals $-\sin A(u)$, (Art. 190).

Similarly, if we change u into $u+4\omega$, the numerator becomes

$$\sin A(u+4\omega)\sin A(u+6\omega)\ldots$$
$$\sin A\left\{u+2(p-1)\omega\right\}\sin A(u)\sin A(u+2\omega).$$

Hence it appears that if in the expression for y, we substitute successively $u+2\omega$, $u+4\omega$, &c. in the place of u, the same series of sines perpetually recurs, and the same value results, abstracting the change of sign. If, therefore, we make $u=\omega$, 3ω, 5ω, &c., the numerator will by each substitution become equal to the denominator, bating the sign; and y will $=+1$, -1, alternately; and if we make $u=0$, 2ω, 4ω, &c., then y will vanish at each substitution, for one of the factors of the numerator will become $\sin A(2np\omega)=0$.

Again, suppose $u=\omega-x$, x being less than ω, then grouping the factors in pairs from the beginning and end, the numerator may be written

$$\sin A(\omega-x)\sin A\left\{(2p-1)\omega-x\right\}$$
$$\times\sin A(3\omega-x)\sin A\left\{(2p-3)\omega-x\right\}\times\&c.,$$
$$\text{or, Art. 190,}\quad\sin A(\omega-x)\sin A(\omega+x)$$
$$\times\sin A(3\omega-x)\sin A(3\omega+x)\times\&c.;$$

to which must be annexed the single factor $\sin A(p\omega-x)$, when p is an odd number. Now all the partial products have the same value whether x be positive or negative, and are all greatest when $x=0$, (Art. 191.); therefore y has the same value and the same sign when u is at equal distances from 0 and 2ω, and attains its greatest value 1, when $u=\omega$. The same thing may be proved when $u=3\omega-x$, $5\omega-x$, &c. Hence we conclude that y represents the sine of an angle ψ, which increases continually with $A(u)$ or ϕ, and attains the values

$$0,\ \frac{\pi}{2},\ 2\frac{\pi}{2},\ 3\frac{\pi}{2},\ \&c.,$$

at the same time that $\phi=0$, a_1, a_2, a_3, &c.; and which has but one value, between $m\frac{\pi}{2}$ and $(m+1)\frac{\pi}{2}$, for any given value of ϕ between a_m and a_{m+1}.

195. We shall now put our assumed value of y into a shape analogous to that which we know the value of $\sin\psi$ or $\sin p\phi$, in Art. 193, can be made to assume, viz.:

$$\sin\psi = p\sin\phi\left\{1-(\sin\phi\operatorname{cosec}2a)^2\right\}$$
$$\left\{1-(\sin\phi\operatorname{cosec}4a)^2\right\}\ \ldots\ \left\{1-[\sin\phi\operatorname{cosec}(p-1)a]^2\right\},$$

p being odd (for it becomes necessary to distinguish between the cases of p even and p odd). In the case then of p an odd number, there will be an even number of factors in the numerator of y after the first, which may be grouped two and two, thus:

$$\sin A(u+2\omega)\sin A\left\{u+2(p-1)\omega\right\}$$
$$\times\sin A(u+4\omega)\sin A\left\{u+2(p-2)\omega\right\}\times\&c.$$
$$\times\sin A\left\{u+(p-1)\omega\right\}\sin A\left\{u+(p+1)\omega\right\},$$
$$\text{or,}\quad -\sin A(u+2\omega)\sin A(u-2\omega)$$
$$\times -\sin A(u+4\omega)\sin A(u-4\omega)\times\&c.$$
$$\times -\sin A\left\{u+(p-1)\omega\right\}\sin A\left\{u-(p-1)\omega\right\}.$$

Hence, making $n=2$, 4, &c. in the formula of Art. 191, the numerator is transformed into

$$\sin\phi\,\frac{(\sin a_2)^2-(\sin\phi)^2}{1-(k\sin\phi\sin a_2)^2}$$

$$\times\frac{(\sin a_4)^2-(\sin\phi)^2}{1-(k\sin\phi\sin a_4)^2}\cdots\frac{(\sin a_{p-1})^2-(\sin\phi)^2}{1-(k\sin\phi\sin a_{p-1})^2}.$$

Also, since $\sin a_{2r+1}=\sin a_{2p-2r-1}$, grouping the factors two and two as before, and omitting the middle factor $\sin a_p=1$, the denominator may be written

$$(\sin a_1\sin a_3\ldots\sin a_{p-2})^2.$$

Hence, making $\beta=\left(\dfrac{\sin a_2\sin a_4\ldots\sin a_{p-1}}{\sin a_1\sin a_3\ldots\sin a_{p-2}}\right)^2$,

and putting $x=\sin\phi$, we have

$$y=\beta x\frac{1-(x\operatorname{cosec} a_2)^2}{1-(kx\sin a_2)^2}\,\frac{1-(x\operatorname{cosec} a_4)^2}{1-(kx\sin a_4)^2}\cdots\frac{1-(x\operatorname{cosec} a_{p-1})^2}{1-(kx\sin a_{p-1})^2},$$

which is the value of $\sin\psi$, and, we observe, coincides with the above expression for $\sin p\phi$, when $k=0$.

196. The next step is to deduce the value of $\cos\psi$.

Let the above equation be written $y=\beta x\,\dfrac{P}{R}$,

P and R representing rational functions, each of $\dfrac{p-1}{2}$ dimensions in x^2;

$$\therefore\ 1-y^2=\frac{R^2-(\beta xP)^2}{R^2}.$$

The numerator is a rational function of p dimensions in x^2, and will vanish whenever $y^2=1$, i. e. by Art. 194, whenever $x^2=(\sin a_{2n+1})^2$, $2n+1$ being any odd number less than $2p$; the numerator therefore is divisible by

$$\{1-(x\operatorname{cosec} a_1)^2\}\,\{1-(x\operatorname{cosec} a_3)^2\}\ldots\{1-(x\operatorname{cosec} a_{2p-1})^2\},$$

$$\text{or } \left\{\text{since } \sin a_{2n+1} = \sin a_{2p-2n-1}, \text{ and } \sin a_p = 1\right\},$$

$$\{1-(x \operatorname{cosec} a_1)^2\}^2\{1-(x \operatorname{cosec} a_3)^2\}^2 \dots \{1-(x \operatorname{cosec} a_{p-2})^2\}^2(1-x^2).$$

But this expression is of p dimensions in x^2, and the term not involving x^3 is 1, therefore it is identical with the numerator;

$$\therefore\ 1 - y^2 =$$

$$\frac{1-x^2}{R^2}\{1-(x \operatorname{cosec} a_1)^2\}^2\{1-(x \operatorname{cosec} a_3)^2\}^2 \dots \{1-(x \operatorname{cosec} a_{p-2})^2\}^2$$

$$= (1 - x^2)\frac{Q^2}{R^2} \text{ suppose;}$$

$$\text{hence } \cos\psi = \cos\phi\,\frac{Q}{R}, \text{ and } \tan\psi = \beta\tan\phi\,\frac{P}{Q}.$$

197. The next step of the proof would be to substitute $\beta x \dfrac{P}{R}$ for y in the equation

$$\frac{d_x y}{\sqrt{(1-y^2)(1-h^2y^2)}} = \beta\,\frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

which arises from $F_h(\psi) = \beta F_k(\phi)$ by differentiation, and to see whether by giving to the indeterminate modulus h a proper value, the two sides could be made identical. This process is impracticable except for small values of p, such as 3, 5; but the necessity of it is obviated by means of a property of the differential equation, which must be common to all its integrals. The property is, that the equation is satisfied by putting $\dfrac{1}{kx}$ for x and $\dfrac{1}{hy}$ for y, the factor $\sqrt{-1}$ being suppressed in each member; and this without subjecting h to any condition except that of being a constant. Therefore $y = \beta x \dfrac{P}{R}$, if it be a solution, as well as $1 - y^2 = (1-x^2)\dfrac{Q^2}{R^2}$, which expresses the same relation between x and y under a different form, must likewise have the property of being satisfied when x is changed into $\frac{1}{kx}$ and y into $\frac{1}{hy}$, a proper value being assigned to h.

198. We proceed therefore to make these substitutions, and shall thus obtain further relations between x and y, which, combined with those already found, will suffice to shew that the equation is satisfied by $y = \beta x \frac{P}{R}$.

The general factor of y is $\frac{1-(x \operatorname{cosec} \alpha)^2}{1-(kx \sin \alpha)^2}$, which, putting $\frac{1}{kx}$ in place of x, becomes

$$\frac{1}{k^2 (\sin \alpha)^4} \, \frac{1-(kx \sin \alpha)^2}{1-(x \operatorname{cosec} \alpha)^2}; \ \therefore \ \frac{P}{R} \text{ will become } \frac{1}{A} \cdot \frac{R}{P},$$

where $A = k^{p-1} (\sin \alpha_2 \sin \alpha_4 \ldots \sin \alpha_{p-1})^4$; and $y = \beta x \frac{P}{R}$, will become

$$\frac{1}{hy} = \frac{\beta}{kx} \cdot \frac{1}{A} \cdot \frac{R}{P}, \quad \text{or } y = \frac{kA}{\beta^2 h} \cdot \beta x \frac{P}{R};$$

therefore in order that this may coincide with $y = \beta x \frac{P}{R}$, we must have $kA = \beta^2 h$,

$$\text{or } h = k^p (\sin \alpha_1 \sin \alpha_3 \ldots \sin \alpha_{p-2})^4.$$

Again, any factor of $\frac{Q}{R}$, such as $\frac{1-(x \operatorname{cosec} \alpha_1)^2}{1-(kx \sin \alpha_2)^2}$, will become

$$\frac{1}{k^2 (\sin \alpha_1 \sin \alpha_2)^2} \, \frac{1-(kx \sin \alpha_1)^2}{1-(x \operatorname{cosec} \alpha_2)^2}; \ \therefore \ \frac{Q}{R} \text{ will become}$$

$$\frac{k}{\beta h} \cdot \frac{T}{P}, \text{ since } \beta h = k^p (\sin \alpha_1 \sin \alpha_2 \ldots \sin \alpha_{p-1})^2,$$

putting $T = \{1-(kx \sin \alpha_1)^2\} \{1-(kx \sin \alpha_3)^2\} \ldots \{1-(kx \sin \alpha_{p-2})^2\}$.

Hence, the equation

$$1 - y^2 = (1 - x^2)\frac{Q^2}{R^2},$$

will be transformed into

$$1 - \frac{1}{(hy)^2} = \left(1 - \frac{1}{(kx)^2}\right)\frac{k^2}{(\beta h)^2}\,\frac{T^2}{P^2},$$

$$\text{or } 1 - (hy)^2 = \{1 - (kx)^2\}\left(\frac{y}{\beta x}\right)^2\frac{T^2}{P^2} = \{1 - (kx)^2\}\frac{T^2}{R^2}$$

since $y = \beta x\,\frac{P}{R}$; $\therefore$ multiplying these equations together,

$$(1 - y^2)\{1 - (hy)^2\} = (1 - x^2)\{1 - (kx)^2\}\frac{(QT)^2}{R^4}.$$

199. In the above equation, substitute for y its value;

$$\therefore \{R^2 - (\beta xP)^2\}\{R^2 - h^2(\beta xP)^2\} = (1 - x^2)\{1 - (kx)^2\}(QT)^2.$$

Since P, Q, R, T, are all polynomials of $(p-1)$ dimensions in x, the first member of this equation is the product of four polynomials, $R + \beta xP$, $R - \beta xP$, &c. each of p dimensions in x, no two of which have a common divisor, because R and P have no common divisor; each of them therefore is composed of one simple factor, and $\frac{1}{2}(p-1)$ double factors, that being the only supposition which agrees with the form of the second member. Also $(QT)^2$ is equal to the product of all the double factors on the first side multiplied by a constant. But every double factor of the first side, being a double factor of a polynomial of the form $R + cxP$, is a factor of

$Rd_x(xP) - xPd_xR$, $\because$, c being any constant quantity,

$$(R + cxP)\,d_x(xP) - xPd_x(R + cxP) = Rd_x(xP) - xPd_xR;$$

$\therefore$ all the factors of QT are contained in $Rd_x(xP) - xPd_xR$, and these two quantities are of the same dimensions, and the term independent of x in each is unity,

$$\therefore\ QT = R d_x (xP) - xP d_x R = R^2 d_x \left(\frac{xP}{R}\right) = \frac{R^2}{\beta} d_x y;$$

and $QT\sqrt{(1-x^2)\{1-(kx)^2\}} = R^2\sqrt{(1-y^2)\{1-(hy)^2\}}$,

$\therefore$, dividing, $\dfrac{1}{\sqrt{(1-x^2)\{1-(kx)^2\}}} = \dfrac{1}{\beta}\dfrac{d_x y}{\sqrt{(1-y^2)\{1-(hy)^2\}}}$;

$$\therefore\ F_k(\phi) = \frac{1}{\beta} F_h(\psi).$$

Hence it is demonstrated, that the assumed value of $\sin\psi$ effects the transformation, in the case of p being any odd number.

200. The equation of the amplitudes admits of a remarkable transformation, by which the calculation of ψ is greatly facilitated.

We have $\sin\psi = \beta\sin\phi\,\dfrac{P}{R}$, $\cos\psi = \cos\phi\,\dfrac{Q}{R}$;

now every factor of P and Q is of the form

$1 - (\sin\phi\operatorname{cosec}\alpha)^2$ which $= (\cos\phi)^2\{1 - (\tan\phi\cot\alpha)^2\}$,

and every factor of R is of the form

$1-(k\sin\phi\sin\alpha)^2$, which $= (\cos\phi)^2[1+\{1-(k\sin\alpha)^2\}(\tan\phi)^2]$
$= (\cos\phi)^2[1+\{\Delta_k(\alpha)\tan\phi\}^2]$,

and there are $\frac{1}{2}(p-1)$ factors in each of P, Q, R; if therefore we put $\tan\phi = z$, we shall have

$P = (\cos\phi)^{p-1}\{1-(z\cot\alpha_2)^2\}\{1-(z\cot\alpha_4)^2\}$
$\quad\ldots\{1-(z\cot\alpha_{p-1})^2\} = (\cos\phi)^{p-1}M$ suppose,

$Q = (\cos\phi)^{p-1}\{1-(z\cot\alpha_1)^2\}\{1-(z\cot\alpha_3)^2\}$
$\quad\ldots\{1-(z\cot\alpha_{p-2})^2\} = (\cos\phi)^{p-1}N$,

$R = (\cos\phi)^{p-1}(1+z^2\Delta_2^2)(1+z^2\Delta_4^2)\ldots(1+z^2\Delta_{p-1}^2)$,

denoting by Δ_2, Δ_4, &c. $\Delta_k(\alpha_2)$, $\Delta_k(\alpha_4)$, &c.;

$$\therefore\ \sin\psi = \frac{\beta\varkappa M}{R}(\cos\phi)^p,\ \ \cos\psi = \frac{N}{R}(\cos\phi)^p;$$

$$\therefore\ N^2 + (\beta\varkappa M)^2 = \frac{R^2}{(\cos\phi)^{2p}} = (1+\varkappa^2)(1+\varkappa^2\Delta_2^2)^2(1+\varkappa^2\Delta_4^2)^2 \ldots (1+\varkappa^2\Delta_{p-1}^2)^2.$$

This equation may be resolved into

$$N+\beta\varkappa M\sqrt{-1} = (1+\varkappa\sqrt{-1})(1+\varkappa\Delta_2\sqrt{-1})^2\ldots(1+\varkappa\Delta_{p-1}\sqrt{-1})^2,$$
$$N-\beta\varkappa M\sqrt{-1} = (1-\varkappa\sqrt{-1})(1-\varkappa\Delta_2\sqrt{-1})^2\ldots(1-\varkappa\Delta_{p-1}\sqrt{-1})^2;$$

for $N+\beta\varkappa M\sqrt{-1}$ cannot have a factor of the form $1-\varkappa\Delta_{2n}\sqrt{-1}$, nor $N-\beta\varkappa M\sqrt{-1}$ one of the form $1+\varkappa\Delta_{2n}\sqrt{-1}$;

therefore dividing the second by the first, and observing that

$$\tan\psi = \beta\tan\phi\,\frac{M}{N},$$

$$\frac{1-\tan\psi\sqrt{-1}}{1+\tan\psi\sqrt{-1}} = \frac{1-\varkappa\sqrt{-1}}{1+\varkappa\sqrt{-1}}\left(\frac{1-\varkappa\Delta_2\sqrt{-1}}{1+\varkappa\Delta_2\sqrt{-1}}\right)^2\left(\frac{1-\varkappa\Delta_4\sqrt{-1}}{1+\varkappa\Delta_4\sqrt{-1}}\right)^2$$
$$\ldots\left(\frac{1-\varkappa\Delta_{p-1}\sqrt{-1}}{1+\varkappa\Delta_{p-1}\sqrt{-1}}\right)^2;$$

therefore taking the logarithms of both sides, and substituting the equivalent angles by the formula

$$\theta = \frac{1}{2\sqrt{-1}}\log_e\left(\frac{1-\tan\theta\sqrt{-1}}{1+\tan\theta\sqrt{-1}}\right),$$

$$\psi = \phi + 2\phi_2 + 2\phi_4 + \text{\&c.} + 2\phi_{p-1};$$

ϕ_{2n} being determined by the equation

$$\tan\phi_{2n} = \tan\phi\,\Delta_{2n} = \tan\phi\sqrt{1-k^2(\sin\alpha_{2n})^2}.$$

201. The theorem has been extended by Mr Ivory to the case when p is an even number. He has shewn (Phil. Trans. 1831.) that in that case, if $y = \sin\psi$, $x = \sin\phi$, and

$$y = \frac{\beta x\sqrt{1 - x^2}}{\sqrt{1 - (kx)^2}}\,\frac{1 - (x \operatorname{cosec} a_2)^2}{1 - (kx \sin a_2)^2}$$

$$\frac{1 - (x \operatorname{cosec} a_4)^2}{1 - (kx \sin a_4)^2} \cdots \frac{1 - (x \operatorname{cosec} a_{p-2})^2}{1 - (kx \sin a_{p-2})^2},$$

$$\text{where } \beta = \left(\frac{\sin a_2 \sin a_4 \ldots \sin a_{p-2}}{\sin a_1 \sin a_3 \ldots \sin a_{p-1}}\right)^2,$$

$$\text{and } h = k^p (\sin a_1 \sin a_3 \ldots \sin a_{p-1})^4,$$

$$\text{then } F_h(\psi) = \beta F_k(\phi).$$

The demonstration is effected by the same steps and methods as for p an odd number; and the equation of the amplitudes may be replaced by

$$\psi = \phi + \phi_0 + \phi_2 + \phi_4 + \&\text{c.} + \phi_{p-2},$$

where

$$\tan\phi_0 = \tan\phi\sqrt{1 - k^2}, \text{ and } \tan\phi_{2n} = \tan\phi\sqrt{1 - (k \sin a_{2n})^2}.$$

202. Hence, $F_k(\phi)$ can be transformed into a similar function, of which the modulus h is less than the given modulus k; and by repeating the process upon $F_h(\psi)$, we can find a second transformed function with a still smaller modulus; and by continuing the operation, we can reduce the proposed function as near as we please to an angle. If we could determine ϕ in terms of ψ, we could transform $F_h(\psi)$ into a similar function $F_k(\phi)$, with a larger modulus; but this would require the solution of an equation of p dimensions; it is to obviate this inconvenience that Jacobi has invented a new transformation, the subject of his second theorem, which we now proceed to give; its enunciation is as follows.

203. If the amplitudes of two functions $F_{k'}(\tau)$, $F_{k'}(\sigma)$, be connected by the equation

$$y = \beta x \frac{1 + (x \cot a_2)^2}{1 + (x \cot a_1)^2}\,\frac{1 + (x \cot a_4)^2}{1 + (x \cot a_3)^2} \cdots \frac{1 + (x \cot a_{p-1})^2}{1 + (x \cot a_{p-2})^2},$$

where $x = \sin\sigma$, $y = \sin\tau$, p is any odd integer, and β, α_1, α_2, &c. are the same as before; and the *complements* of their moduli by the equation

$$h = k^p (\sin\alpha_1 \sin\alpha_3 \dots \sin\alpha_{p-2})^4,$$

the functions themselves are connected by the equation

$$F_{h'}(\tau) = \beta F_{k'}(\sigma).$$

204. We have seen that by the substitution of $\beta x \frac{P}{R}$ for y in the equation

$$\frac{d_x y}{\sqrt{(1-y^2)\{1-(hy)^2\}}} = \beta \frac{1}{\sqrt{(1-x^2)\{1-(kx)^2\}}},$$

the two members are made identical; therefore they will be made identical by all values of x and y, which satisfy the equation $y = \beta x \frac{P}{R}$, whether real or imaginary, provided the constants k, h, β remain the same.

Let, therefore, $x = \sqrt{-1}\tan\sigma$, $y = \sqrt{-1}\tan\tau$,

$$\therefore \beta \frac{d_\tau \sigma}{\sqrt{1-(k'\sin\sigma)^2}} = \frac{1}{\sqrt{1-(h'\sin\tau)^2}},$$

denoting by k', h' the complements of k, h;

$$\therefore \beta F_{k'}(\sigma) = F_{h'}(\tau).$$

Also $\tan\psi = \sqrt{-1}\sin\tau$, $\tan\phi = \sqrt{-1}\sin\sigma$; and by Art. 196.

$$\tan\psi = \beta\tan\phi \frac{1-(\operatorname{cosec}\alpha_2 \sin\phi)^2}{1-(\operatorname{cosec}\alpha_1 \sin\phi)^2}$$

$$\frac{1-(\operatorname{cosec}\alpha_4 \sin\phi)^2}{1-(\operatorname{cosec}\alpha_3 \sin\phi)^2} \dots \frac{1-(\operatorname{cosec}\alpha_{p-1} \sin\phi)^2}{1-(\operatorname{cosec}\alpha_{p-2} \sin\phi)^2},$$

$\therefore$ the relation between the amplitudes becomes

$$\sin\tau = \beta\sin\sigma \frac{1+(\operatorname{cosec}\alpha_2 \tan\sigma)^2}{1+(\operatorname{cosec}\alpha_1 \tan\sigma)^2}, \text{ \&c.}$$

or, since

$$\frac{1 + (\operatorname{cosec} \alpha_2 \tan \sigma)^2}{1 + (\operatorname{cosec} \alpha_1 \tan \sigma)^2} = \frac{(\cos \sigma)^2 + \{1 + (\cot \alpha_2)^2\} (\sin \sigma)^2}{(\cos \sigma)^2 + \{1 + (\cot \alpha_1)^2\} (\sin \sigma)^2}$$

$$= \frac{1 + (\cot \alpha_2 \sin \sigma)^2}{1 + (\cot \alpha_1 \sin \sigma)^2},$$

we have

$$y = \beta x \frac{1 + (x \cot \alpha_2)^2}{1 + (x \cot \alpha_1)^2} \frac{1 + (x \cot \alpha_4)^2}{1 + (x \cot \alpha_3)^2} \cdots \frac{1 + (x \cot \alpha_{p-1})^2}{1 + (x \cot \alpha_{p-2})^2},$$

which gives τ, and shews that τ and σ increase together from 0, and become equal at every multiple of $\frac{\pi}{2}$.

205. Hence, having given $F_{k'}(\sigma)$ to be transformed into a similar function with a larger modulus, we first obtain

$$k = \sqrt{1 - k'^2},$$

and then compute α_1, α_2, &c.; this gives us the value of

$$h = k^p (\sin \alpha_1 \sin \alpha_3 \ldots \sin \alpha_{p-2})^4,$$

and then $h' = \sqrt{1 - h^2}$ is known, which is greater than k', because h is less than k; also the new amplitude τ is known from the above equation, and hence we obtain the new value of the proposed function, viz. $\frac{1}{\beta} F_{k'}(\tau)$; and by continuing the operation, we may raise the proposed function as near as we please to a logarithm. We have supposed p to be an odd number, but by substituting in the formula of Art. (201), we shall easily obtain the formula belonging to the case of p an even number.

206. The comparison of the results of these two transformations

$$\beta F_k(\phi) = F_h(\psi), \quad \beta F_{k'}(\sigma) = F_{h'}(\tau),$$

leads to a remarkable relation between the complete functions.

Since when $\phi = \frac{\pi}{2}$, $\psi = p\frac{\pi}{2}$ (Art. 194), the first equation gives

$$\beta F_k = p F_h.$$

Also, when $\tau = \frac{\pi}{2}$, $\sigma = \frac{\pi}{2}$, therefore the second gives

$$\beta F_{k'} = F_{h'}.$$

Hence, eliminating β, we have

$$F_k F_{h'} = p F_h F_{k'}.$$

207. Again, in the equation $\beta F_{k'}(\sigma) = F_{h'}(\tau)$, change k' into h, and therefore, k into h';

$$\therefore \ \beta' F_h(\sigma) = F_k(\tau),$$

$$\text{where } \beta' = \left(\frac{\sin\mu_2 \sin\mu_4 \dots \sin\mu_{p-1}}{\sin\mu_1 \sin\mu_3 \dots \sin\mu_{p-2}}\right)^2,$$

$$k' = h'^p (\sin\mu_1 \sin\mu_3 \dots \sin\mu_{p-2})^4,$$

μ_1, μ_2, &c., being the amplitudes of $\frac{1}{p}F_{h'}$, $\frac{2}{p}F_{h'}$, &c. and which are also to be substituted for α_1, α_2, &c. in the value of y or $\sin\tau$.

Hence, $\beta' F_h = F_k$, and $\beta F_k = p F_h$, $\therefore \ \beta\beta' = p$.

$$\text{Let } \sigma = \psi, \ \therefore \ F_h(\psi) = \frac{1}{\beta'} F_k(\tau),$$

$$\therefore \ F_k(\phi) = \frac{1}{\beta\beta'} F_k(\tau) = \frac{1}{p} F_k(\tau).$$

Hence, we can find any multiple or aliquot part of an elliptic function. Similar formulæ may be obtained for p an even number.

208. We shall in conclusion, apply the general formulæ of Arts. 192 and 201, to two particular cases.

Let $p = 2$; $\therefore\ \beta = \frac{1}{(\sin \alpha_1)^2} = 1 + k'$, (Art. 160),

$$h = k^2 (\sin \alpha_1)^4 = \frac{1 - k'^2}{(1 + k')^2} = \frac{1 - k'}{1 + k'},$$

$$\sin \psi = \frac{\beta \sin \phi \cos \phi}{\sqrt{1 - (k \sin \phi)^2}} = \frac{(1 + k') \cos \phi \sin \phi}{\sqrt{1 - (k \sin \phi)^2}},$$

which may be transformed into

$$\tan (\psi - \phi) = k' \tan \phi,$$

$$\text{and } F_k(\phi) = \frac{1}{1 + k'} F_h(\psi).$$

Hence we fall upon Lagrange's transformation, these formulæ agreeing with the formulæ of Art. (179).

209. Let $p = 3$, $\therefore\ \beta = \left(\frac{\sin \alpha_2}{\sin \alpha_1}\right)^2 = \frac{2}{\sin \alpha_1} - 1$,

as will be easily seen by eliminating k between the equations

$$\sqrt{1 - k^2} \tan \alpha_2 \tan \alpha_1 = 1, \quad \tan \tfrac{1}{2} \alpha_2 = \tan \alpha_1 \sqrt{1 - k^2 (\sin \alpha_1)^2};$$

$$h = k^3 (\sin \alpha_1)^4,$$

$$\sin \psi = \beta \sin \phi \, \frac{1 - \left(\frac{\sin \phi}{\sin \alpha_2}\right)^2}{1 - (k \sin \phi \sin \alpha_2)^2},$$

which may be transformed into

$$\tan \tfrac{1}{2} (\psi - \phi) = \tan \phi \sqrt{1 - (k \sin \alpha_2)^2} = \left(\frac{1}{\sin \alpha_1} - 1\right) \tan \phi;$$

$$\text{and } F_k(\phi) = \frac{\sin \alpha_1}{2 - \sin \alpha_1} F_h(\psi).$$

Upon this transformation, Legendre had hit, previous to the discoveries of Jacobi; the amplitude α_1 is known from Art. 165.

210. We shall now prove a remarkable relation, discovered by Legendre, between complete functions of the first and second orders, whose moduli are complementary to one another, and which may be useful as a formula of verification; viz., that

$$E_c E_b - G_c G_b = \frac{\pi}{2}, \text{ where } G_c \text{ denotes } E_c - F_c.$$

We have $d_c E_c(\phi) = \int_\phi \frac{-c(\sin\phi)^2}{\Delta} = \frac{1}{c}\int_\phi \frac{\Delta^2 - 1}{\Delta} = \frac{1}{c} G_c(\phi)$,

$$\therefore\ d_c E_c = \frac{1}{c} G_c.$$

Similarly, $d_c E_b = d_c b \,.\, d_b E_b = -\frac{c}{b} \cdot \frac{1}{b} G_b = -\frac{c}{b^2} G_b$.

Again, $d_c F_c(\phi) = \int_\phi \frac{c(\sin\phi)^2}{\Delta^3} = \frac{1}{c}\int_\phi \frac{1}{\Delta^3} - \frac{1}{c} F_c(\phi)$

$= \frac{1}{c}\left\{\frac{1}{b^2} E_c(\phi) - \frac{c^2}{b^2}\frac{\sin\phi\cos\phi}{\Delta}\right\} - \frac{1}{c} F_c(\phi)$, by Ex. 2, Art. 154,

$$\therefore\ d_c F_c = \frac{1}{b^2 c}(E_c - b^2 F_c) = \frac{1}{b^2 c}\{c^2 E_c + b^2 (E_c - F_c)\} = \frac{c}{b^2} E_c + \frac{1}{c} G_c;$$

$$\text{hence } d_c G_c = d_c E_c - d_c F_c = -\frac{c}{b^2} E_c;$$

$$\text{similarly, } d_c G_b = -\frac{c}{b} \cdot -\frac{b}{c^2} E_b = \frac{1}{c} E_b.$$

Now let C denote a function of c, and suppose

$$C = E_c E_b - G_c G_b,$$

$$\therefore\ d_c C = E_b \cdot \frac{1}{c} G_c + E_c\left(-\frac{c}{b^2} G_b\right) - G_b \,.\, d_c G_c - G_c \,.\, d_c G_b$$

$$= G_c\left(\frac{1}{c} E_b - d_c G_b\right) - G_b\left(\frac{c}{b^2} E_c + d_c G_c\right) = 0,$$

therefore C does not involve c, and is the same for all values of c.

Now if c be very small, $F_c = \frac{\pi}{2}\left(1 + \frac{c^2}{4}\right)$, $E_c = \frac{\pi}{2}\left(1 - \frac{c^2}{4}\right)$, $F_b = \log_\epsilon\left(\frac{4}{c}\right)$, $E_b = 1$. Therefore, since $E_bF_c - (F_c - E_c)F_b = C$, we have $\frac{\pi}{2} - \frac{\pi c^2}{4}\log_\epsilon\left(\frac{4}{c}\right) = C$; but when $c = 0$, $c^2 \log_\epsilon\left(\frac{4}{c}\right) = 0$;

$$\therefore\ C = \frac{\pi}{2};\quad \therefore\ E_cE_b - G_cG_b = \frac{\pi}{2},$$

$$\text{or } E_cF_b + E_bF_c - F_cF_b = \frac{\pi}{2}.$$

211. The equations

$$d_cE_c = \frac{1}{c}(E_c - F_c),\quad d_cF_c = \frac{1}{b^2c}(E_c - b^2F_c),$$

give $F_c = E_c - c \,.\, d_cE_c$, $E_c = (1 - c^2)(F_c + c \,.\, d_cF_c)$, by means of which the value of either of the complete functions F_c, E_c, can be deduced from the known value of the other.

Differentiating the former of the above equations, we find $d_cF_c = -c \,.\, d_c^2E_c$; therefore, substituting in the latter,

$$E_c = (1 - c^2)\left\{E_c - c \,.\, d_cE_c - c^2 \,.\, d_c^2E_c\right\},$$

$$\text{or } (1 - c^2) \,.\, d_c^2E_c + \frac{1 - c^2}{c} \,.\, d_cE_c + E_c = 0.$$

Similarly, differentiating the latter, and substituting the values thence obtained of E_c and d_cE_c, in the former,

$$(1 - c^2) \,.\, d_c^2F_c + \frac{1 - 3c^2}{c} \,.\, d_cF_c - F_c = 0.$$

If we regard E_c and F_c as functions of b, the complement of the modulus, since

$$d_c E_c = -\frac{c}{b} \cdot d_b E_c, \quad d_c^2 E_c = \frac{c^2}{b^2} \cdot d_b^2 E_c - \frac{1}{b^3} \cdot d_b E_c,$$

by substitution, we find

$$(1 - b^2) \cdot d_b^2 E_c - \frac{1 + b^2}{b} \cdot d_b E_c + E_c = 0;$$

$$\text{similarly,} \quad (1 - b^2) \cdot d_b^2 F_c + \frac{1 - 3b^2}{b} \cdot d_b F_c - F_c = 0.$$

Hence the complete function of the first order is determined by the same differential equation, whether it be regarded as a function of the modulus, or of the complement of the modulus.

212. We next come to the consideration of the function of the third order

$$\Pi_c(n, \phi) = \int_\phi \frac{1}{1 + n(\sin\phi)^2} \frac{1}{\sqrt{1 - (c\sin\phi)^2}},$$

in which enters a new element, viz., the parameter n capable of all values from $-\infty$ to $+\infty$; this can always be made to depend upon a similar function, having the same modulus and amplitude and a parameter between 0 and -1, as we proceed to shew, after Legendre.

We shall first shew that the parameter may be always supposed less than the modulus.

Let there be two functions of the third order

$$\int_\phi \frac{1}{\{1 + n(\sin\phi)^2\}\Delta}, \text{ and } \int_\phi \frac{1}{\left\{1 + \frac{c^2}{n}(\sin\phi)^2\right\}\Delta},$$

having the same modulus and amplitude c and ϕ, but different parameters, viz. n and $\frac{c^2}{n}$, whose product is equal to c^2.

Now $$\frac{1}{1+n(\sin\phi)^2}+\frac{1}{1+\frac{c^2}{n}(\sin\phi)^2}-1$$

$$=\frac{1-c^2(\sin\phi)^4}{1+\left(n+\frac{c^2}{n}\right)(\sin\phi)^2+c^2(\sin\phi)^4}$$

$$=\frac{1-c^2(\sin\phi)^4}{(\cos\phi)^2\{1-c^2(\sin\phi)^2\}+(\sin\phi)^2(1+n+\frac{c^2}{n}+c^2)}$$

$$=\frac{1-c^2(\sin\phi)^4}{(\cos\phi)^2\Delta^2+(1+n)\left(1+\frac{1}{n}c^2\right)(\sin\phi)^2}$$

$$=\frac{1}{\Delta^2}\frac{(\sec\phi)^2-c^2(\sin\phi)^2\{(\sec\phi)^2-1\}}{1+a\left(\frac{\tan\phi}{\Delta}\right)^2},$$

$$=\Delta\frac{d_\phi\left(\frac{\tan\phi}{\Delta}\right)}{1+a\left(\frac{\tan\phi}{\Delta}\right)^2},\text{ making } a=(1+n)\left(1+\frac{1}{n}c^2\right);$$

therefore, dividing by Δ, and integrating,

$$\Pi_c(n,\phi)+\Pi_c\left(\frac{c^2}{n},\phi\right)=F_c(\phi)+\frac{1}{\sqrt{a}}\tan^{-1}\left(\frac{\sqrt{a}\tan\phi}{\Delta_c(\phi)}\right).$$

Hence a function containing a parameter $>c$, can always be made to depend upon one whose parameter is $<c$.

213. The above equation furnishes immediately the value of $\Pi_c(c,\phi)$ or $\Pi_c(-c,\phi)$, by making $n=c$, or $=-c$.

Also if we make $a=0$, and, therefore, $n=-1$, or $=-c^2$,

we have $$\Pi_c(-1,\phi)+\Pi_c(-c^2,\phi)=F_c(\phi)+\frac{\tan\phi}{\Delta_c(\phi)}.$$

And if we make $n = c(\cos\theta + \sqrt{-1}\sin\theta)$, we have

$$\frac{c^2}{n} = c(\cos\theta - \sqrt{-1}\sin\theta), \text{ and } a = 1 + 2c\cos\theta + c^2,$$

and so obtain the value of the integral

$$\int_\phi \frac{1 + c\cos\theta(\sin\phi)^2}{1 + 2c\cos\theta(\sin\phi)^2 + c^2(\sin\phi)^4} \cdot \frac{1}{\Delta_c(\phi)}.$$

214. We have supposed a to be positive, and therefore n either to be positive, or to be negative and intermediate to $-c^2$ and -1, that is, to be of one of the forms

$$(\cot\theta)^2, \text{ or } -1 + b^2(\sin\theta)^2.$$

When n is negative, and not intermediate to $-c^2$ and -1, that is, of one of the forms $-(\operatorname{cosec}\theta)^2$, or $-c^2(\sin\theta)^2$, and consequently a negative, and $= -a_1$ suppose, the last term of the above equation must be replaced by the logarithmic function

$$\frac{1}{2\sqrt{a_1}} \log_\epsilon \frac{\Delta + \sqrt{a_1}\tan\phi}{\Delta - \sqrt{a_1}\tan\phi}.$$

These two latter cases of the parameter, by the preceding equation, depend one upon the other, since their product $= c^2$; that is, a function with a negative parameter not intermediate to 0 and $-c^2$, can always be made to depend upon a similar function with the same modulus and amplitude, and parameter between these limits.

215. We shall next shew that the two first cases of the parameter can be made to depend one upon the other; that is, that a function with a positive parameter, can be made to depend upon a similar function with the same modulus and amplitude, and parameter intermediate to $-c^2$ and -1.

Let there be two parameters n and $-m$ so related that

$$(1 + n)(1 - m) = b^2, \text{ or } m - n + mn = c^2;$$

$$\text{Now } \frac{c^2+n}{1+n(\sin\phi)^2}+\frac{c^2-m}{1-m(\sin\phi)^2}-c^2$$

$$=\frac{c^2+n-m-2mn(\sin\phi)^2+mnc^2(\sin\phi)^4}{1+(n-m)(\sin\phi)^2-mn(\sin\phi)^4}$$

$$=mn\,.\,\frac{1-2(\sin\phi)^2+c^2(\sin\phi)^4}{1+(mn-c^2)(\sin\phi)^2-mn(\sin\phi)^4}$$

$$=mn\,.\,\frac{(\cos\phi)^2-(\sin\phi)^2\Delta^2}{\Delta^2+mn(\sin\phi\cos\phi)^2}$$

$$=mn\,\frac{\left\{\dfrac{\cos\phi}{\Delta}\right\}^2-(\sin\phi)^2}{1+mn\left\{\dfrac{\cos\phi\sin\phi}{\Delta}\right\}^2}$$

$$=mn\Delta\,\frac{d_\phi\left\{\dfrac{\cos\phi\sin\phi}{\Delta}\right\}}{1+mn\left\{\dfrac{\cos\phi\sin\phi}{\Delta}\right\}^2};$$

therefore, dividing by $mn\Delta$, and integrating,

$$\frac{c^2+n}{mn}\,\Pi_c(n,\phi)+\frac{c^2-m}{mn}\,\Pi_c(-m,\phi)$$

$$=\frac{c^2}{mn}\,F_c(\phi)+\frac{1}{\sqrt{mn}}\tan^{-1}\left\{\frac{\sqrt{mn}\cos\phi\sin\phi}{\Delta_c(\phi)}\right\},$$

$$\text{or } \frac{1+n}{n}\,\Pi_c(n,\phi)-\frac{1-m}{m}\,\Pi_c(-m,\phi)$$

$$=\frac{c^2}{mn}\,F_c(\phi)+\frac{1}{\sqrt{mn}}\tan^{-1}\left\{\frac{\sqrt{mn}\cos\phi\sin\phi}{\Delta_c(\phi)}\right\};$$

no constant being added, because every term vanishes when $\phi=0$.

Now let $n = (\cot\theta)^2$ in the equation of condition

$$(1+n)(1-m) = b^2,$$

$$\therefore\ 1 - m = b^2(\sin\theta)^2, \text{ or } -m = -1 + b^2(\sin\theta)^2;$$

hence a function with a positive parameter, can be made to depend upon a similar function with the same modulus and amplitude, and parameter between $-c^2$ and -1.

In the case of complete functions, substituting for m and n the above values, we find

$$(\sec\theta)^2\Pi_c(n) - \left\{\frac{b\sin\theta}{\Delta_b(\theta)}\right\}^2\Pi_c(-m) = \left\{\frac{c\tan\theta}{\Delta_b(\theta)}\right\}^2 F_c,$$

$$\text{or } \Pi_c(n) = \left\{\frac{\sin\theta\cos\theta}{\Delta_b(\theta)}\right\}^2\{b^2\Pi_c(-m) + c^2(\sec\theta)^2 F_c\}.$$

216. Hence it results from the preceding formulæ, that an elliptic function of the third order $\Pi_c(n, \phi)$, having for its parameter any real quantity n, can always be made to depend upon a similar function $\Pi_c(m, \phi)$ having the same modulus and amplitude, and of which the parameter m is in every case between 0 and -1; that is,

1. between -1 and $-c^2$, or of the form $-1 + b^2(\sin\theta)^2$, when n is positive;
2. between 0 and $-c^2$, or of the form $-c^2(\sin\theta)^2$, when n is negative.

Also since the function of the parameter $(1+n)\left(1+\dfrac{c^2}{n}\right)$, which we have denoted by a, is made positive by the first form of the parameter, and negative by the second, it follows that functions with these respective parameters are essentially different, and not reducible one to the other; functions of the former sort in their reduction and comparison involving angles, and those of the latter, logarithms. The first of the above fundamental forms of the parameter may for the sake of distinction be called *circular*, and the second form *logarithmic*.

217. To investigate a formula for the comparison of elliptic functions of the third order.

If σ be the given amplitude of a function of the third order, and ϕ and ψ two other amplitudes subject to the condition

$$\cos\sigma = \cos\phi\cos\psi - \sin\phi\sin\psi\sqrt{1 - c^2(\sin\sigma)^2},$$

$$\text{then } \Pi_c(n, \phi) + \Pi_c(n, \psi) - \Pi_c(n, \sigma)$$

$$= \frac{1}{\sqrt{a}}\tan^{-1}\frac{n\sqrt{a}\sin\phi\sin\psi\sin\sigma}{1 + n - n\cos\phi\cos\psi\cos\sigma}.$$

Since ψ is a function of ϕ, we may assume

$\Pi_c(n, \phi) + \Pi_c(n, \psi) - \Pi_c(n, \sigma) = P$ (a function of ϕ);

$$\therefore \frac{1}{1 + n(\sin\phi)^2}\cdot\frac{1}{\Delta(\phi)} + \frac{1}{1 + n(\sin\psi)^2}\cdot\frac{d_\phi\psi}{\Delta(\psi)} = d_\phi P,$$

$$\text{or, since } \frac{1}{\Delta(\phi)} + \frac{d_\phi\psi}{\Delta(\psi)} = 0,$$

$$\frac{1}{\Delta(\phi)}\left\{\frac{1}{1 + n(\sin\phi)^2} - \frac{1}{1 + n(\sin\psi)^2}\right\} = d_\phi P,$$

$$\text{or } \frac{n}{\Delta(\phi)}\cdot\frac{(\sin\psi)^2 - (\sin\phi)^2}{1 + n(\sin\phi)^2 + n(\sin\psi)^2 + (n\sin\phi\sin\psi)^2} = d_\phi P.$$

But $\Delta(\phi) + \Delta(\psi)d_\phi\psi = c^2 d_\phi(\sin\phi\sin\psi\sin\sigma)$,

$$\therefore \{\Delta(\phi)\}^2 - \{\Delta(\psi)\}^2 = c^2(\sin\psi)^2 - c^2(\sin\phi)^2$$

$$= c^2\Delta(\phi)\,d_\phi(\sin\phi\sin\psi\sin\sigma);$$

$$\therefore \frac{n d_\phi(\sin\phi\sin\psi\sin\sigma)}{1 + n(\sin\phi)^2 + n(\sin\psi)^2 + (n\sin\phi\sin\psi)^2} = d_\phi P.$$

Let $\sin\phi\sin\psi\sin\sigma = \frac{1}{x}$, then observing that

$$(\sin\phi)^2 + (\sin\psi)^2 = (\sin\sigma)^2 - 2\sin\phi\sin\psi\cos\sigma\,\Delta(\sigma)$$

$$+\, c^2(\sin\phi\sin\psi\sin\sigma)^2,$$

$$\frac{-n}{z^2\left\{1+n(\sin\sigma)^2\right\} - 2nz\cot\sigma\,\Delta(\sigma) + nc^2 + n^2(\operatorname{cosec}\sigma)^2} = d_z P;$$

$$\therefore\ \frac{1}{\sqrt{a}}\cot^{-1}\frac{z\left\{1+n(\sin\sigma)^2\right\} - n\cot\sigma\,\Delta(\sigma)}{n\sqrt{a}} = P,$$

by Ex. 4. Art. 26; making, as before,

$$a = (1+n)\left(1+\frac{1}{n}c^2\right),$$

and no constant being added, because both sides vanish when

$$\phi = 0, \text{ or } z = \infty, \text{ and } \psi = \sigma.$$

Hence restoring the values of z and P, and reducing by means of the equation of condition,

$$\Pi_c(n, \phi) + \Pi_c(n, \psi) - \Pi_c(n, \sigma)$$

$$= \frac{1}{\sqrt{a}}\tan^{-1}\frac{n\sqrt{a}\sin\phi\sin\psi\sin\sigma}{1+n-n\cos\phi\cos\psi\cos\sigma}.$$

If a be negative, that is, if the parameter be logarithmic, the second member must be replaced by a logarithm.

218. Hence the excess of the sum of the functions whose amplitudes are ϕ and ψ over that whose amplitude is σ, which is nothing in functions of the first order, and algebraic in those of the second, is in functions of the third order expressed by an angle, or by a logarithm if a be negative. From the above equation, we can deduce every thing relating to the comparison of elliptic functions of the third order, in the same manner as we have done for functions of the first and second orders, from their corresponding equations; thus

let $\sigma = \frac{\pi}{2}$, $\therefore\ \Pi_c(n, \phi) + \Pi_c(n, \psi) - \Pi_c(n)$

$$= \frac{1}{\sqrt{a}} \tan^{-1} \frac{n\sqrt{a} \sin\phi \sin\psi}{1+n}, \text{ where } b \tan\phi \tan\psi = 1;$$

and if $\phi = \psi$, and therefore $(\sin\phi)^2 = \frac{1}{1+b}$,

$$2\Pi_c(n, \phi) - \Pi_c(n) = \frac{1}{\sqrt{a}} \tan^{-1} \frac{n\sqrt{a}}{(1+n)(1+b)}.$$

219. To prove that *complete* functions of the third order can always be expressed by functions of the first and second orders.

We have $\Pi_c(n) = \int_\phi \frac{1}{1+n(\sin\phi)^2} \cdot \frac{1}{\Delta_c(\phi)}$,

the integral being taken between the limits $\phi = 0$, $\phi = \frac{\pi}{2}$;

$$\therefore\ d_n \Pi_c(n) = \int_\phi \frac{-(\sin\phi)^2}{\{1+n(\sin\phi)^2\}^2 \Delta_c(\phi)}$$

$$= -\frac{1}{n} \int_\phi \frac{1+n(\sin\phi)^2 - 1}{\{1+n(\sin\phi)^2\}^2 \Delta_c(\phi)},$$

$$\text{or } n \,.\, d_n\Pi_c(n) + \Pi_c(n) = \int_\phi \frac{1}{\{1+n(\sin\phi)^2\}^2 \Delta_c(\phi)}.$$

But by Ex. 3, Art. 154, taking the integrals between the limits $\phi = 0$, $\phi = \frac{\pi}{2}$,

$$\frac{2a}{n} \int_\phi \frac{1}{\{1+n(\sin\phi)^2\}^2 \Delta_c(\phi)} = -\frac{c^2}{n^2} F_c + \frac{1}{n}(E_c - F_c)$$

$$+ \left(1 + \frac{2}{n}(1+c^2) + \frac{3c^2}{n^2}\right) \Pi_c(n),$$

where $a = (1+n)\left(1 + \frac{1}{n} c^2\right)$. Hence, combining this equation with the preceding,

$$2ad_n \Pi_c(n) + \left(1 - \frac{c^2}{n^2}\right) \Pi_c(n) = - \frac{c^2}{n^2} F_c + \frac{1}{n}(E_c - F_c),$$

$$\text{or } 2ad_n \Pi_c(n) + \Pi_c(n)\, d_n a = - \frac{c^2}{n^2} F_c + \frac{1}{n}(E_c - F_c),$$

$$\text{since } d_n a = 1 - \frac{c^2}{n^2}.$$

Hence, if a be positive, dividing by $2\sqrt{a}$, and integrating, we have

$$\sqrt{a} \cdot \Pi_c(n) = - \frac{c^2 F_c}{2} \int_n \frac{1}{n^2 \sqrt{a}} + \frac{1}{2}(E_c - F_c) \int_n \frac{1}{n\sqrt{a}}.$$

But if a be negative and $= -a_1$, then

$$-2a_1 d_n \Pi_c(n) - \Pi_c(n)\, d_n a_1 = - \frac{c^2}{n^2} F_c + \frac{1}{n}(E_c - F_c);$$

therefore, dividing by $-2\sqrt{a_1}$, and integrating as before,

$$\sqrt{a_1} \cdot \Pi_c(n) = \frac{c^2 F_c}{2} \int_n \frac{1}{n^2 \sqrt{a_1}} - \frac{1}{2}(E_c - F_c) \int_n \frac{1}{n\sqrt{a_1}}.$$

In order to effect the integrations, we must consider separately the two fundamental forms of the parameter, the first of which we know makes a positive, and the second makes a negative.

220. CASE I. Let the parameter be circular; and for the sake of convenience, let $n = (\cot\theta)^2$; we can afterwards adapt the formula to the fundamental form of the circular parameter, viz.

$$-1 + b^2 (\sin\theta)^2:$$

$$\therefore\ d_\theta n = -2\cot\theta\,(\operatorname{cosec}\theta)^2,\ \sqrt{a} = \frac{\Delta_b(\theta)}{\cos\theta\sin\theta};$$

$$\therefore\ -\frac{c^2}{2}\int_n \frac{1}{n^2\sqrt{a}} = -\frac{c^2}{2}\int_\theta \frac{d_\theta n}{n^2\sqrt{a}} = \int_\theta \frac{c^2(\tan\theta)^2}{\Delta_b(\theta)}$$

$$= \tan\theta\,\Delta_b(\theta) - E_b(\theta),$$

$$\text{and } \int_n \frac{1}{n\sqrt{a}} = -2\int_\theta \frac{1}{\Delta_b(\theta)} = -2F_b(\theta);$$

$$\therefore\ \sqrt{a}\,.\,\Pi_c(n) = F_c\{\tan\theta\,\Delta_b(\theta) - E_b(\theta)\}$$
$$- F_b(\theta)\,(E_c - F_c) + \text{const.}$$

To determine the constant, let $c = 0$, and therefore $b = 1$;

$$\therefore\ \Delta_b(\theta) = \cos\theta,\ E_b(\theta) = \sin\theta,\ E_c = F_c = \frac{\pi}{2},\ \sqrt{a} = \operatorname{cosec}\theta,$$

$$\text{and } \Pi_{c=0}(n) = \int_\phi \frac{1}{1 + (\cot\theta\sin\phi)^2} = (\sin\theta)^2\int_\phi \frac{(\sec\phi)^2}{(\sin\theta^2)+(\tan\phi)^2}$$

$= \sin\theta\tan^{-1}\left(\frac{\tan\phi}{\sin\theta}\right) = \sin\theta\,.\,\frac{\pi}{2}$, between the limits $\phi=0$, $\phi=\frac{\pi}{2}$;

therefore, constant $= \frac{\pi}{2}$;

$$\therefore\ \frac{\Delta_b(\theta)}{\sin\theta\cos\theta}\,\Pi_c(n) = \frac{\pi}{2} + F_c\{\tan\theta\,\Delta_b(\theta) - E_b(\theta)\}$$
$$- F_b(\theta)\,(E_c - F_c).$$

Now let $-m = -1 + b^2(\sin\theta)^2$, $n = (\cot\theta)^2$; therefore by Art. 215,

$$\frac{\Delta_b(\theta)}{\sin\theta\cos\theta}\,\Pi_c(n) = \frac{\sin\theta\cos\theta}{\Delta_b(\theta)}\{b^2\,\Pi_c(-m) + c^2(\sec\theta)^2 F_c\};$$

hence, substituting and reducing,

$$\frac{b^2\sin\theta\cos\theta}{\Delta_b(\theta)}\{\Pi_c(-m) - F_c\} = \frac{\pi}{2} - F_cE_b(\theta) - (E_c - F_c)\,F_b(\theta).$$

221. CASE II. Let the parameter be logarithmic, or

$$n = -c^2 (\sin\theta)^2,$$

$$\therefore\ d_\theta n = -2c^2 \sin\theta \cos\theta, \text{ and } a = -a_1 = -\{\cot\theta\,\Delta_c(\theta)\}^2;$$

$$\therefore\ \frac{c^2}{2}\int_n \frac{1}{n^2\sqrt{a_1}} = -\int_\theta \frac{1}{(\sin\theta)^2\Delta_c(\theta)}$$

$$= -F_c(\theta) + E_c(\theta) + \Delta_c(\theta)\cot\theta,$$

$$\text{and } -\tfrac{1}{2}\int_n \frac{1}{n\sqrt{a_1}} = -\int_\theta \frac{1}{\Delta_c(\theta)} = -F_c(\theta);$$

$$\therefore\ \cot\theta\,\Delta_c(\theta)\,\Pi_c(n) = F_c\{-F_c(\theta) + E_c(\theta) + \Delta_c(\theta)\cot\theta\}$$

$$- F_c(\theta)(E_c - F_c) + C,$$

$$= F_c\{E_c(\theta) + \Delta_c(\theta)\cot\theta\} - E_c F_c(\theta) + C,$$

and $C = 0$, as is easily shewn by making $c = 0$;

$$\therefore\ \Pi_c(n) = F_c + \frac{\tan\theta}{\Delta_c(\theta)}\{F_c E_c(\theta) - E_c F_c(\theta)\}.$$

The above formulæ are of great importance, for since they reduce complete functions of the third order to functions of the first and second orders, and since in the solution of mechanical and geometrical questions which depend upon this theory, it is usually only complete functions that are required; the necessity of having tables of elliptic functions of the third order, is in a great measure obviated.

222. A function whose parameter is imaginary, and of the form $\gamma(\cos\theta + \sqrt{-1}\sin\theta)$, will, in any real expression whose integral is required, be accompanied by its conjugate function whose parameter is $\gamma(\cos\theta - \sqrt{-1}\sin\theta)$; and their sum, which will be an integral of the form

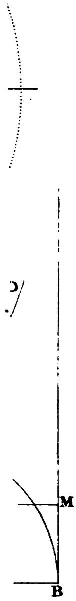
$$\int_\phi \frac{\alpha + \beta(\sin\phi)^2}{1 + 2\gamma\cos\theta(\sin\phi)^2 + \gamma^2(\sin\phi)^4}\cdot\frac{1}{\Delta_c(\phi)},$$

can always be expressed by two other functions, whose parameters are real, one of the form $-1+b^2(\sin\mu)^2$, and the other of the form $-c^2(\sin\lambda)^2$. For the proof of this, and of several other important propositions, recourse must be had to the *Traité des Fonctions Elliptiques* of Legendre, (who is the author of this branch of the Integral Calculus); to Jacobi's *Fundamenta nova theoriæ functionum ellipticarum;* and to Abel's papers in Crelle's Journal.

ERRATA.

Page 15, Ex. 5, line 2, *interchange* b and a.
45, line 20, *for* $N_1 x_1$ *read* $N_1 x$.
84, Ex. 2, *supply* $\int_x$ *twice.*
175, line 4, *for* $\sqrt{a^2-\rho^2+(a^2-c^2)}$ *read* $\sqrt{a^2-\rho^2}+(a^2-c^2)$.
205, last line, *for* $E(\mu)$ *read* $E(\sigma)$.
206, line 6, *for* $\sin\psi$ *read* $\sin\phi$.

Plate 1.



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